Optimal nondiscriminatory auctions with favoritism[1]

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Abstract

In many auction settings, there is favoritism: the seller's welfare depends positively on the utility of a subset of potential bidders. However, laws or regulations may not allow the seller to discriminate among bidders. We find the optimal nondiscriminatory auction in a private value, single-unit model under favoritism. At the optimal auction there is a reserve price, or an entry fee, which is decreasing in the proportion of preferred bidders and in the intensity of the preference. Otherwise, the highest-valuation bidder wins.

Keywords: auctions, favoritism, nondiscriminatory mechanisms.

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1

1 Introduction

It is frequently the case when auctions are used that the seller is not indifferent as to which of the bidders will be the winner. A seller uses an auction to enhance competition among bidders but, for a given selling price, she would prefer some of the bidders to win rather than others. This may occur when some of the bidders' welfare positively influences the seller's welfare. For example, in a government-run auction, domestic firms may generate more tax revenue than their foreign rivals. Alternatively, the seller and some of the bidders may be firms in the same conglomerate. We say that there is favoritism when the seller has such a preference for some bidders over others.

Favoritism usually motivates the design of discriminatory auctions.[5] Since the bidders' identities are relevant to the seller, the rules of the auction are specified in such a way that not only the bids matter, but also who makes them. For example, price preferences are frequently introduced: to win, a non-preferred bidder may have to beat the highest bid made by a preferred bidder by at least some previously specified margin. Another usual way to discriminate, known as *right of first refusal*, is giving one of the preferred bidders the right to match the highest bid that any of her rivals may submit.[6]

However, in many situations discrimination is not possible. This happens quite often in public procurement, where laws and regulations sometimes forbid favoring some bidders over others to level the field and thus to foster competition. There may be higher-level regulations that explicitly prevent local authorities from favoring local firms. In general, this constraint may be interpreted as one imposed by a principal on an agent who is in charge of the auction.[7]

Our aim here is to examine the auction design problem faced by a seller who places positive

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[5]The justification for discrimination that we examine, which derives from the fact that the seller values some of the bidders' utilities, is not the only possible one. With a fixed number of bidders, biasing the auction against strong bidders raises revenue, as shown by optimal auction theory. With endogenous entry, discriminating against strong bidders may also be optimal for the seller, since it could encourage the entry of weak bidders. Those arguments require asymmetry among bidders, while our model is (but for the possibly unequal consideration of bidders' utilities by the seller) symmetric.

[6]This right has been studied in Walker (1999), Bikhchandani et al. (2005), Arozamena and Weinschelbaum (2006), Lee (2008).Burguet and Perry (2009) and Choi (2009).

[7]Violating this constraint can result in unwanted scrutiny and other costs for auction designers. See, for example, Hu (2010) for criticism of the Texas Lottery Commission for perceived conflict of interest in its organizing the procurement auction for Texas' state lottery vendors. A more complete analysis would embed the auction design problem within a principal-agent model, but here we are simply concerned with the effects of a no-discrimination constraint on auction design under seller favoritism among bidders.

weight on some of the bidders' welfare, but faces a no-discrimination constraint. We conclude that the seller will choose an auction where the highest-valuation bidder wins unless her valuation is too low. Thus, there will be a reserve price or an entry fee adequately chosen to exclude bidders with lower valuations, just as in the standard, revenue-maximizing auction. However, we find that the set of valuations excluded is smaller when there is favoritism: there will be a lower reserve price or entry fee. Furthermore, the set of excluded valuations becomes smaller when the weight attached to the utility of any favored bidder grows. Hence, favoritism raises the probability of selling the object.

Our work contributes to the literature on favoritism in auctions. Laffont and Tirole (1991) and Vagstad (1995) study the case of multidimensional auctions, where favoritism may appear when the auctioneer assesses product quality. McAfee and McMillan (1989), Branco (1994), and Naegelen and Mougeot (1998) examine single-dimensional auctions, where price-preferences may be used. The main result of this literature is that the optimal allocation rule follows from comparing the maximum valuation among preferred bidders with the maximum "virtual" valuation among non-preferred bidders. In all these papers, the seller is allowed to use discriminatory mechanisms. Arozamena and Weinschelbaum (2011) extend the analysis of the single-dimensional case to a situation where the number of bidders is endogenous, and conclude that the optimal auction in that setting is indeed nondiscriminatory. Thus, not being allowed to discriminate is irrelevant in that setting. Here, however, we examine the case where the number of bidders is fixed.

In the following section we present the model and the basic results.

2   The model and the optimal mechanism

The owner of a single, indivisible object is selling it through an auction.[8] For simplicity, we assume the seller attaches no value to the object. There are $N \geq 2$ bidders whose valuations for the object are given by v_i, $i = 1, ..., N$. Each v_i is bidder i's private information. These valuations are distributed identically and independently according to the c.d.f. F with support on the interval $[\underline{v}, \overline{v}]$ and a density f that is positive and bounded on the whole support. The context is therefore one of symmetric independent private values. All parties to the auction are risk-neutral, and we assume that the virtual valuation of any bidder, $J(v) = v - \frac{1-F(v)}{f(v)}$, is increasing in her actual valuation.[9]

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[8] All of our results, however, are applicable as well to the case of procurement auctions.

[9] This is what the literature calls the "regular case" after Myerson (1981).

Our aim is to characterize a selling mechanism that maximizes the utility of a seller who values positively the welfare of a subset of the set of bidders, in addition to her own expected revenue. Specifically, we assume that the seller's objective function follows from adding to the seller's revenue each bidder's welfare, where bidder i's welfare is weighted according to an exogenous parameter α_i, $i = 1, ..., N$. We assume as well that $\alpha_i \in [0, 1]$ for all i. That is, the seller attaches a weakly positive weight to each bidder's welfare, but cannot value the latter more than her own, "private" utility (i.e., her revenue). Note that if $\alpha_i = 0$ for all i, we have a standard, revenue-maximizing seller. However, if $\alpha_i = 1$ for all i, the seller will behave as she would when pursuing efficiency in the absence of favoritism.

As described so far, our problem is a slight modification of the standard optimal auction problem with independent private values.[10] Let $H_i(v_1, ..., v_N)$ $(P_i(v_1, ..., v_N))$ be the probability that bidder i gets the object (respectively, the price bidder i has to pay to the seller) if bidder valuations are given by $(v_1, ..., v_N)$. Then, the seller has to choose a mechanism $\{H_i(.), P_i(.)\}_{i=1}^{N}$ such that, for all $(v_1, ...v_N)$, $0 \leq H_i(v_1, ..., v_N) \leq 1$ for all i and $\sum_{i=1}^{N} H_i(v_1, ..., v_N) \leq 1$. In addition, let $h_i(v_i)$ $(p_i(v_i))$ be the expected probability that bidder i gets the object (respectively, the expected price she pays) when her valuation is v_i, and the valuations of all other bidders are unknown.

Bidder i's expected utility when her valuation is v_i and she announces that it is v_i' is

$$\widetilde{U}_i(v_i, v_i') = h_i(v_i')v_i - p_i(v_i').$$

Additionally, let

$$U_i(v_i) = \widetilde{U}_i(v_i, v_i) = h_i(v_i)v_i - p_i(v_i).$$

Then, the seller's problem is

$$\max_{\{H_i(.), P_i(.)\}_{i=1}^{N}} \sum_{i=1}^{N} \left(\int_{\underline{v}}^{\overline{v}} p_i(v_i)f(v_i)dv_i + \alpha_i \int_{\underline{v}}^{\overline{v}} U_i(v_i)f(v_i)dv_i \right)$$

subject to the standard incentive compatibility and participation constraints

$$U_i(v_i) \geq \widetilde{U}_i(v_i, v_i') \quad \text{for all } i, \text{ for all } v_i, v_i'$$
$$U_i(v_i) \geq 0 \quad\quad\quad\quad\quad \text{for all } i, \text{ for all } v_i.$$

This problem has been studied before, for instance, in Naegelen and Mougeot (1998).[11] However, as mentioned above, we are interested in the case where the seller *cannot* discriminate

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[10] See Myerson (1981) and Riley and Samuelson (1981).

[11] It can be thought of as an extension to the N-bidder context of a particular case of the analysis in Naegelen and Mougeot (1998), when there is no consumer surplus and the shadow cost of public funds is zero.

among bidders. Hence, we add a new constraint on the set of mechanisms $\{H_i(.), P_i(.)\}_{i=1}^{N}$ that the seller can select.

No-discrimination: *The seller has to choose a mechanism* $\{H_i(.), P_i(.)\}_{i=1}^{N}$ *that, for any permutation* $\pi : \{1, ..., N\} \longrightarrow \{1, ..., N\}$, *satisfies*

$$
\begin{aligned}
H_i(v_{\pi(1)}, ..., v_{\pi(N)}) &= H_{\pi(i)}(v_1, ..., v_N) \\
P_i(v_{\pi(1)}, ..., v_{\pi(N)}) &= P_{\pi(i)}(v_1, ..., v_N)
\end{aligned}
$$

for all i.

Once discrimination is ruled out, we must have

$$
h_i(v_i) = h(v_i) \text{ and } p_i(v_i) = p(v_i) \text{ for all } i.
$$

In deriving the optimal auction, we follow the usual steps in the literature. Let $\widetilde{v}_i(v_i)$ be the valuation that bidder i announces optimally when her true valuation is v_i. Clearly, by incentive compatibility, it has to be true that $\widetilde{v}_i(v_i) = v_i$ and $U_i(v_i) = \widetilde{U}_i(v_i, \widetilde{v}_i(v_i))$. The envelope theorem then implies that

$$
U_i'(v_i) = \frac{\partial}{\partial v_i}\widetilde{U}_i(v_i, \widetilde{v}_i(v_i)) = h(v_i).
$$

Therefore, $U_i(v_i) = \int_{\underline{v}}^{v_i} h(s)ds + U_i(\underline{v})$. Stated in a way that is more convenient for us in what follows, and noting that, in the solution to our problem, $U_i(\underline{v}) = 0$ for all i,[12] we have

$$
p(v_i) = h(v_i)v_i - \int_{\underline{v}}^{v_i} h(s)ds
$$

for all i. Substituting for $p_i(v_i)$ and $U_i(v_i)$ in the seller's objective function yields

$$
\sum_{i=1}^{N} \left[\int_{\underline{v}}^{\overline{v}} \left[h(v_i)v_i - \int_{\underline{v}}^{v_i} h(s)ds \right] f(v_i)dv_i + \alpha_i \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v_i} h(s)ds f(v_i)dv_i \right].
$$

Integrating by parts, we have

$$
\sum_{i=1}^{N} \int_{\underline{v}}^{\overline{v}} h(v_i) \left[v_i - (1 - \alpha_i) \frac{1 - F(v_i)}{f(v_i)} \right] f(v_i)dv_i.
$$

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[12] $U_i(\underline{v})$ may be zero or positive for those bidders i with $\alpha_i = 1$. Given that we are adding the expected utilities of the seller and these bidders, how much they pay (as long as incentive compatibility holds) does not affect the seller's objective function. However, by no-discrimination, $U_i(v_i) = U(v_i)$ for all i, so $U(\underline{v}) > 0$ is only possible if $\alpha_i = 1$ for all i. But even in this case there exists a solution where $U_i(\underline{v}) = 0$ for all i.

Then, the seller solves

$$\max_{\{H_i(.)\}_{i=1}^N} E_{v_1,...,v_N}\left[\sum_{i=1}^N H_i(v_i)\left[v_i - (1-\alpha_i)\frac{1-F(v_i)}{f(v_i)}\right]\right]$$

subject to the no-discrimination constraint.

The fact that the seller cannot discriminate among bidders, however, allows us to restate this problem in a more convenient way.

Remark 1 *Let $v_{(n)}$ be the nth order statistic associated with the vector of valuations $(v_1,...,v_N)$.*[13] *For any vector $\mathbf{v} = (v_1,...,v_N)$, we may define a function $\gamma_{\mathbf{v}} : \{1,...,N\} \longrightarrow \{1,...,N\}$ that assigns to each position $1,...,N$ the identity of the bidder whose valuation ranks in that position. That is, $\gamma_{\mathbf{v}}(n) = i$ if $v_{(n)} = v_i$.*[14] *The no-discrimination constraint implies that for any two vectors $\mathbf{v} = (v_1,...,v_N)$, $\mathbf{v}' = (v_1',...,v_N')$ such that $(v_{(1)},...,v_{(N)}) = (v_{(1)}',...,v_{(N)}')$ we must have*

$$H_{\gamma_{\mathbf{v}}(n)}(v_1,...,v_N) = H_{\gamma_{\mathbf{v}'}(n)}(v_1',...,v_N'), \ n = 1,...,N.$$

To show this, assume $H_{\gamma_{\mathbf{v}}(n^*)}(v_1,...,v_N) \neq H_{\gamma_{\mathbf{v}'}(n^*)}(v_1',...,v_N')$ for some n^*. Let $\widetilde{\pi} : \{1,...,N\} \longrightarrow \{1,...,N\}$ be such that $\widetilde{\pi}(i) = \gamma_{\mathbf{v}}(\gamma_{\mathbf{v}'}^{-1}(i))$. The functions $\gamma_{\mathbf{v}}$ and $\gamma_{\mathbf{v}'}$ are permutations, so $\widetilde{\pi}$ is a permutation as well. Note that $(v_1',...,v_N') = (v_{\widetilde{\pi}(1)},...,v_{\widetilde{\pi}(N)})$. Then,

$$H_{\widetilde{\pi}(i^*)}(v_1,...,v_N) \neq H_{i^*}(v_{\widetilde{\pi}(1)},...,v_{\widetilde{\pi}(N)})$$

for $i^* = \gamma_{\mathbf{v}'}(n^*)$, which violates the no-discrimination constraint.

Thus, if for any two vectors of valuations the corresponding vectors of order statistics coincide, then the seller has to allocate the good with the same probability to those bidders that occupy each ordered position in the vectors of order statistics. In other words, the probability that any given bidder wins must depend only on the vector of order statistics and on her valuation's position in that vector. This, in turn, implies the following lemma.

Lemma 1 *The seller's problem can be expressed in terms of order statistics: she has to choose an allocation function*

$$\{H_n(v_{(1)},...,v_{(N)})\}_{n=1}^N.$$

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[13] We follow the convention by which $v_{(1)}$ is the highest value in the vector, $v_{(2)}$ the second-highest, and so on.

[14] Since we are using continuous distributions, ties will occur with probability zero. Still, in the event of a tie, positions have to be allocated with equal probabilities among those bidders whose valuations coincide so that the no-discrimination constraint is satisfied.

That is, all valuation vectors that generate the same vector of order statistics have to be treated equally. Then, we can focus only on which allocations the seller chooses when the vector of valuations is ordered. Allocations in all other cases follow from the no-discrimination constraint.

The seller, though, cares about the identities of the bidders. Given a vector of order statistics $(v_{(1)}, ..., v_{(N)})$, since valuations are independently drawn from the same distribution, the probability that bidder i's valuation ranks in position n is the same for all bidders. Then, for that vector of order statistics, the seller's objective function will take the following expected value

$$\sum_{n=1}^{N} \frac{1}{N} H_n(v_{(1)}, ..., v_{(N)}) \left[\sum_{i=1}^{N} \left[v_{(n)} - (1 - \alpha_i) \frac{1 - F(v_{(n)})}{f(v_{(n)})} \right] \right]$$

or,

$$\sum_{n=1}^{N} \frac{1}{N} H_n(v_{(1)}, ..., v_{(N)}) \left[N v_{(n)} - \frac{1 - F(v_{(n)})}{f(v_{(n)})} \sum_{i=1}^{N} (1 - \alpha_i) \right].$$

The seller's problem is then

$$\max_{\{H_n(v_{(1)}, ..., v_{(N)})\}_{n=1}^{N}} E_{v(1), ..., v(N)} \left\{ \sum_{n=1}^{N} \frac{1}{N} H_n(v_{(1)}, ..., v_{(N)}) \left[N v_{(n)} - \frac{1 - F(v_{(n)})}{f(v_{(n)})} \sum_{i=1}^{N} (1 - \alpha_i) \right] \right\}.$$

The solution to this problem is simple. Since $J(v) = v - \frac{1-F(v)}{f(v)}$ is increasing in v, it is easy to show that $Nv - \frac{1-F(v)}{f(v)} \sum_{i=1}^{N} (1 - \alpha_i)$ is also increasing in v. Thus, the seller should allocate the object with probability 1 to the bidder with the highest valuation whenever $N v_{(1)} - \frac{1-F(v_{(1)})}{f(v_{(1)})} \sum_{i=1}^{N} (1 - \alpha_i) > 0$. Otherwise, she should keep the object. We therefore have the following result.

Proposition 1 *The optimal allocation rule is*[15]

$$H_i(v_1, ..., v_N) = \begin{cases} 1 \ \ if \ v_i > \max_{j \neq i} v_j \ \ and \ \ N v_i - \frac{1-F(v_i)}{f(v_i)} \sum_{j=1}^{N} (1 - \alpha_j) > 0 \\ \\ 0 \hspace{6cm} otherwise. \end{cases}$$

This direct mechanism can be implemented by any auction where the highest-valuation bidder wins, with an adequately chosen reserve price or entry fee. For example, the seller may choose a first-price or a second-price auction with reserve price r such that

$$Nr - \frac{1 - F(r)}{f(r)} \sum_{i=1}^{N} (1 - \alpha_i) = 0.[16]$$

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[15] In order to satisfy the no-discrimination constraint, as we mentioned above, if there is a tie all bidders with the highest valuation win with the same probability.

[16] As the left-hand side of this equation is increasing in r, there is a unique solution.

Note that if $\alpha_i = 0$ for all i, the optimal mechanism for the seller described in Proposition 1 coincides with the standard, revenue-maximizing direct mechanism: the object is awarded to the highest-valuation bidder and all bidders with valuations below r such that $r - \frac{1-F(r)}{f(r)} = 0$ are excluded. At the same time, if $\alpha_i = 1$ for all i, then $r = 0$, no bidders are excluded and the seller chooses an auction where the highest-valuation bidder always wins.

Therefore, for any vector of weights $(\alpha_1, ..., \alpha_N)$ that the seller attaches to the bidders' utilities, she chooses a mechanism that neither sells the good with probability one nor attains revenue-maximization. Furthermore, she selects a mechanism that falls in between these two extreme cases.[17]

Example 1 *Consider the case of two bidders with valuations drawn from the Uniform distribution on the unit interval with seller favoritism for bidder 1 given by $\alpha_1 = 1/2$ and no favoritism for bidder 2 so that $\alpha_2 = 0$. It is well-known that the standard, revenue-maximizing reserve price in this case is $r = 1/2$. Taking into consideration the seller's favoritism toward bidder 1 and her inability to discriminate between the bidders, the optimal mechanism's reserve price is $r = 3/7$. Finally, the reserve price that ensures that the good is sold with probability one is $r = 0$.*

Note as well that, as long as $\alpha_i > 0$ for some i (i.e., there is favoritism), the optimal mechanism's reserve price may increase when the number of bidders changes. The optimal reserve price depends on $(1/N) \sum_{i=1}^{N} \alpha_i$. Then, if a new bidder enters the auction with a weight in the seller's objective function that is higher (lower) than the average weight that the seller attaches to the existing bidders' utilities, the optimal reserve price will fall (respectively, rise). This is in contrast to the standard, revenue-maximizing reserve price, which is independent of N.

It is also interesting to examine, given N, the effect of a change in the vector of weights on the mechanism selected by the seller and on the welfare of each of the parties involved in the auction. First, notice that r, the minimum valuation that is not excluded from the mechanism, is decreasing in α_i for any i. If the seller places a larger weight on a given bidder's welfare, the only instrument she has to enhance that bidder's welfare is to reduce the reserve price or entry fee that she employs in any auction that implements the optimal mechanism. Doing so

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[17]When the number of bidders is endogenous, as Arozamena and Weinschelbaum (2011) show, even with favoritism it is optimal for the seller to use a nondiscriminatory mechanism which maximizes revenue and at the same time sells the good with probability one.

benefits not only the bidder whose corresponding weight has risen, but all other bidders as well. Therefore, all bidders' expected utilities are increasing in any α_i.

It is not the case, though, that the seller is "sharing" her gains from having a higher α_i. There are actually two effects. First, for any given value of r, the seller's utility straightforwardly grows with α_i. Second, the seller increases her utility by reducing r. This second effect raises the utilities of all bidders, too.

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