# Analytical Results on a Decentralized Combinatorial Auction (Extended Abstract)

S. Damla Ahipaşaoğlu   Richard Steinberg
Department of Management, London School of Economics

March 29, 2011

## Abstract

**We provide analytical results on a decentralized combinatorial auction; specifically, the PAUSE Auction Procedure. We prove that the auctioneer's revenue under PAUSE is at least as great as the revenue under the VCG mechanism when there are two bidders, and provide lower bounds on the revenue under PAUSE when there are an arbitrary number of bidders.**

## 1 Basic Results

In this paper, we compare the multistage combinatorial bidding procedure PAUSE (as described in Kelly and Steinberg, [1], and Land, Powell, and Steinberg, [2]) versus the well-known VCG mechanism with regard to auctioneer's revenue. Denote the set of $n$ bidders by $\mathcal{B} := \{B_1, B_2, ..., B_n\}$, the set of $m$ items by $\Omega$, and the value to bidder $B_i$ of a subset of items $S \subseteq \Omega$ by $v_i(S)$. We assume that the bidders have privately known values where these values are superadditive, i.e., $v_i(A \cup B) \geq v_i(A) + v_i(B)$ for any bidder $B_i$ and any $A, B \subset \Omega$ such that $A \cap B = \emptyset$. Further, we assume that the bidders are individually rational and engage in straightforward bidding.

PAUSE is a progressive auction that is comprised of $m$ stages, each of which has a finite number of rounds. Stage 1 is run as a simultaneous ascending auction in which bidders submit increasing bids on individual items in each round until no bidder increases his bid. To bid in Stage $k$, for $k \geq 2$, a bidder is required to submit a *composite bid*, which is a partition of the set of items into nonempty subsets called *blocks*, where each block has a specific bidder assigned to it—either the bidder submitting the composite bid, or another bidder in the auction—together with a bid price. The triple consisting of the block, the bidder on the block, and the bid price are collectively called the *block bid*. When constructing a composite bid in stage $k$, each bidder is subject to the restriction that the number of items in each block can be at most $k$.

The sum of all bid prices in the composite bid is called the *evaluation* of the composite bid. In each round, the composite bids submitted are *validated* by the auctioneer. A composite bid is valid if it satisfies the conditions above in addition to the *improvement margin requirement*, which requires the evaluation to be at least $\epsilon$

higher than the evaluation of the last standing composite bid, where $\epsilon > 0$ defined by the auctioneer. At the end of each round, the auctioneer announces a standing composite bid, which has the highest evaluation among the valid bids (ties are broken randomly). Each stage terminates with a round during which no valid composite bid is submitted. The standing composite bid at the end of Stage $m$ determines the final allocation and prices. We denote by $\Pi^P$ and $\Pi^{VCG}$ the revenue, respectively, of the auctioneer under PAUSE and the VCG mechanism. We start our analysis with the simplest scenario, in which there are only two bidders, and obtain the result that PAUSE weakly dominates the VCG mechanism with regard to revenue generation.

**Proposition 1** *PAUSE auction generates almost at least as much revenue as the VCG mechanism when there are two bidders. Specifically:*

$$\Pi^P \;\geq\; \Pi^{VCG} - 2\epsilon. \tag{1.1}$$

We generalize the previous result in Proposition 2, which will constitute an important building block for our further results.

**Proposition 2** *Consider any auction instance. Then:*

$$\Pi^P \;\geq\; \frac{1}{(n-1)^2}\Pi^{VCG} - \frac{n}{n-1}\epsilon. \tag{1.2}$$

## 2 Main Results

We next show that, by taking into account the individual item prices at the end of the PAUSE auction, the result of Proposition 2 can be strengthened.

**Proposition 3** *Let $\rho_\omega^P$ be the final price of item $\omega \in \Omega$ in the PAUSE auction. Then:*

$$\Pi^P \;\geq\; \frac{1}{(n-1)^2}\Pi^{VCG} + \frac{n(n-2)}{(n-1)^2}\sum_{\omega\in\Omega}\rho_\omega^P - \frac{n}{n-1}\epsilon. \tag{2.3}$$

Next, we will focus on auctions with specific structure. In particular, we first look at the scenario where there is a bidder who values the combination of all items in the auction more than the total value of these items to any subset of the rest of the bidders.

**Definition 1** *A bidder $B_j$ is referred to as a* dominant bidder *if $v_j(\Omega) = \arg\max_{(X_1,\ldots,X_n)\in\mathcal{P}} \sum_{i\in\mathcal{B}} v_i(X_i)$, where $\mathcal{P}$ is the set of all partitions of $\Omega$.*

**Proposition 4** *Consider an auction instance for which there exists a dominant bidder. Then:*

$$\Pi^P \;\geq\; \frac{1}{n-1}\Pi^{VCG} - n\epsilon. \tag{2.4}$$

*Further, this bound is sharp.*

In the remainder of the paper, we consider two other specific scenarios. First, the situation where for each item the values of the two highest bidders are close to each other. Second, where the synergies are bounded. We make use of the following definitions and, based on the discussion below, arrive at Proposition 6.

**Definition 2** *For each item $\omega \in \Omega$, let $(i_1, i_2, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$ such that $v_{i_1}(\omega) \geq v_{i_2}(\omega) \ldots \geq v_{i_n}(\omega)$. Then we say that there is* $\alpha$-competition *for item $\omega$ if the value functions of the bidders satisfy $v_{i_2}(\omega) \geq \alpha v_{i_1}(\omega)$.*

**Definition 3** *A bidder $B_i$ is said to have $\delta$-bounded synergies if, for all $S \subseteq \Omega$, $v_i(S) \leq (1+\delta)\sum_{\omega \in S} v_i(\omega)$.*

We can find constants $\alpha$ and $\delta$ that apply for all items $\omega \in \Omega$ and all bidders $B_i \in \mathcal{B}$, respectively. These constants are independent of the auction at hand and satisfy $\alpha \in [0,1]$ and $\delta \geq 0$. There can be many constants that satisfy these definitions ($\alpha = 0$ and $\delta = \infty$ for example); clearly we are interested in the largest value of $\alpha$ and the smallest value of $\delta$. We should not be surprised that the ratio $\gamma := (1+\delta)/\alpha$ plays a role in bounding the revenue obtainable in a combinatorial auction. We show in the results below that this is in fact the case. We call $\gamma$ the *bounding factor*.

**Proposition 5** *Consider a combinatorial auction instance in which all bidders have $\delta$-bounded synergies and there is $\alpha$-competition for all items. Let $\Pi$ be the revenue generated by a combinatorial auction on this instance. Also let $\rho_\omega^{SAA}$ be the final price of item $\omega \in \Omega$ in a simultaneous ascending auction. Then:*

$$\Pi \;\leq\; \gamma \sum_{\omega \in \Omega} \rho_\omega^{SAA}, \tag{2.5}$$

*where $\gamma := (1+\delta)/\alpha$.*

Since the prices for individual items at the end of a PAUSE auction must be at least as great as the prices in a SSA auction minus $\epsilon$, we have the following corollary to Proposition 5:

**Corollary** *Let $\rho_\omega^P$ be the final price of item $\omega \in \Omega$ in a PAUSE auction. If all bidders have $\delta$-bounded synergies and there is at least $\alpha$-competition for all items, then:*

$$\Pi^{VCG} \;\leq\; \gamma \left( \sum_{\omega \in \Omega} \rho_\omega^P + \epsilon |\Omega| \right). \tag{2.6}$$

This corollary leads us to our final proposition.

**Proposition 6** *If all bidders have $\delta$-bounded synergies and there is at least $\alpha$-competition for all items, then:*

$$\Pi^P \;\geq\; \frac{\gamma + n(n-2)}{(n-1)^2 \gamma} \Pi^{VCG} - \left( \frac{n}{n-1} + \frac{n(n-2)|\Omega|}{(n-1)^2} \right) \epsilon. \tag{2.7}$$

**Corollary** *As the number of bidders increases, the bound in Proposition 6 converges to*

$$\Pi^P \;\geq\; \frac{1}{\gamma} \Pi^{VCG} - \epsilon. \tag{2.8}$$

The paper will conclude with a discussion of future research directions.

## References


[1] Kelly, F. and Steinberg, R., A combinatorial auction with multiple winners for universal service. *Management Science* 46(4): 586-596, 2000.

[2] Land, A., Powell, S., and Steinberg, R., PAUSE: A computationally tractable combinatorial auction. In: Cramton, P.C., Shoham, Y., and Steinberg, R., (eds.) *Combinatorial auctions*, MIT Press, Cambridge, MA, USA, pp. 139-157, 2006.