Pairwise stability in graduate college admission problem with budget constraints when students are picky.

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Abstract

We study graduate college admission problem with budget constraint. Each college has a fixed amount of money to distribute as stipends among a set of students matched to it. Also, each college has additively separable preferences over the set of students and has a nonnegative value for each student. On the other hand, each student is matched with at most one college and receives a stipend from it. Each student has quasi-linear preferences over college-stipend bundles.

In this paper, we consider fixed budget (feasibility) constraint for college admission problem which was not studied in earlier literature. We define pairwise stability and show that a pairwise stable allocation always exists. We introduce a rule through an algorithm we construct, which always selects a pairwise stable allocation.

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Keywords: Quasi-linear preferences, additively separable preferences, pairwise stability, budget constraint.

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1 Introduction

We study graduate college admission problem. There are college graduates applying for graduate studies. Also, there are colleges that have graduate programs. Each such college wants to admit certain number of students. In most colleges, at the beginning of the year, department allocates a fixed budget for graduate student admissions. With this budget, admission committee can offer stipends to students that it wants to admit. At the beginning of each year, admission committee agrees on certain amount as the maximum stipend it can offer to a student. No student will receive a stipend higher than this amount. Money left after admissions goes back to department for other uses. Therefore admission committee has no preference over the money it will be left with at the end. Each college values each student differently. Decision of a college on which student to admit, depends on the value of the student to the college. Each student, on the other hand, makes decisions that depends on which college offers him admission and with what stipend. Pursuing graduate study is not mandatory. Therefore, each student always has an outside option which maybe staying at home. Each student has the lowest stipend that he would like to receive from each college in order to consider the admission from this college as an option. This amount can be different for each college. If a college offers a stipend which is less than this lowest amount, then the student would prefer to stay at home rather than going to this college with offered stipend. We match students to colleges and allocate the budgets of the college among students as stipends. We seek for an allocation with the following property: consider a pair of a student and a college that are not matched to each other. They cannot come together and find a stipend such that the student will be willing to go to this college with this stipend and the college will be better off by admitting this student.

The college admission problem was first studied by Gale and Shapley (1962) in their seminal paper where they propose the well-known deferred-acceptance algorithm. When the preferences of the colleges are responsive to the preferences over individual students, the deferred-acceptance algorithm gives a core allocation. It shows that the core is not empty for this model (Roth (1984, 1985), Roth and Sotomayor (1990) and etc.). When we introduce money to this problem, under certain assumptions on the preferences, competitive allocation, which coincides with the core for this model, exists (Shapley and Shubik (1971), Crawford and Knoer (1981), Crawford and Kelso (1982), Quinzii (1984), Gerard van der Laan, Talman and Tang (1997), Sun and Yang (2006)). Some of the works in the literature consider the problem where budget of a college is flexible and determined by total productivity (Crawford and Knoer (1981), Crawford and Kelso (1982), Sotomayor (1992, 1999, 2002, 2007, 2009), Sun and Yang (2006)). Different from earlier papers, in our model we assume that colleges have fixed budgets. This means that, the budget does not depend on how many students each college admits and the identity of students. Fixed budget is generally the case in graduate college admissions: in many graduate programs, department allocates fixed amount of money for graduate student admissions and this amount doesn't change until the next year.

Recently matching problem with general contracts was studied. Sufficient conditions on preferences for existence of stable allocation were provided (Hatfield and Milgrom (2005), Hatfield and Kojima (2008, 2010)). These works generalize many papers in the matching literature stated above. We will show with an example that, conditions provided in these papers that are sufficient for existence of stable allocation, do not encompass budget constraint. This makes our results independent of the results in those papers.

One different assumption of our paper is that, colleges have cardinal preferences over students, which is the value of student for it. Colleges have no preference over money. In many graduate programs, department allocates certain amount of money for admissions. Admission committee can only use this money to admit students. Money left at the end of admissions will go back to department. Therefore, assuming that admission committee has no preference over money is realistic.

In most of the graduate programs, admission committee agrees on the maximum stipend that they can offer to a student. No student will receive a stipend higher than this amount. This is the case in real life: it is very unlikely for a college to offer a student unlimited stipend. Each college generally sets the maximum amount that it can offer and does not offer stipends higher than that amount.

As in many papers in the literature, we also assume that colleges have additively separable preferences. In other words, the value of a college for a student is independent of the identity of other students admitted to this college.

As we mentioned earlier, each student has a preference over a college that he is admitted to and a stipend he receives from it. We assume that preferences of students are quasi-linear. This assumption may seem restrictive, but weakening this assumption would result in many complications. We let each student have an outside option which may be staying at home.

With this option, each student has the lowest stipend that he would like to receive from each college in order to consider it as an option. This amount can be different for each college. When a college offers a stipend below this amount, student prefers staying at home rather than going to that college. Assumption of outside option makes our model more general. A special case would be assuming that students do not have any other option and would like to go to all colleges, even if they do not receive any stipends from them.

We define a notion of stability called pairwise stability. This property says the following: Let an allocation be selected. Suppose there is college A and student B, such that student B is not assigned to college A. Also suppose that there is student C assigned to college A whom college values less than student B. Therefore, college A would be willing to release student C if it can admit student B using the stipend it is paying to student C (Note that, college A can release more than one student, in order to admit student B, as long as the value of student B is higher than the total value of this group of students for college A). Suppose that college A and student B can agree on a stipend such that student B will prefer college A with this stipend to his initial allocation. As we mentioned above, college A will be better off by admitting this student even if it has to release student C. For an allocation to be pairwise stable there should be no such deviations by a pair of college and student. Our main result is that the pairwise stable allocation always exits. We construct an algorithm and prove that the rule associated with this algorithm always selects a pairwise stable allocation.

Pairwise stability may seem as a weaker requirement compared to coalitional stability which is immune to deviations by a group of students and a college. Nevertheless, pairwise deviations are the ones that are more likely to happen in real life: A student may contact a college he is not assigned to with a proposal: in case this college could offer him admission with certain stipend, he would be willing to go to this college. Then the college considers whether it can benefit from this proposal and makes a decision accordingly. Reverse proposal can also be the case. Our requirement is that no such deviation by a college and a student should be beneficial for both of them. In real life, it is very unlikely that two or more students who are assigned to different colleges can communicate and agree with some other college and move to that college. Therefore, by considering real life applications we can see that our requirement is pretty strong. The rest of the paper is organized as follows. In Section 2 we define the model and pairwise stability and in Section 3 we provide our algorithm. In section 4 we state our main result and provide the proof.

2 Model

There are a finite set of **colleges** $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ and a finite set of **students** $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$. Each student has the option of staying at home. We denote this option $\emptyset_c$. Each college c has **capacity** $q_c \in \mathbb{Z}_+$ and the null college has $q_{\emptyset_c} = \infty$. Let $q \equiv (q_{c_1}, q_{c_2}, \ldots, q_{c_n}) \in \mathbb{Z}^n_+$ be the capacity profile. Each college c has a fixed **budget** $B^c \in \mathbb{R}_+$ that it can distribute as stipend to the students it admits. Let $B \equiv \{B^{c_1}, B^{c_2}, \ldots, B^{c_n}\} \in \mathbb{R}^n_+$ be the budget profile. Each college c sets an amount m^c to be the **maximal stipend** they can award to any student. Let $m = \{m^{c_1}, m^{c_2}, \ldots, m^{c_m}\}$ be the maximal stipend profile. Welfare of college c over sets of students can be represented by a **utility function** $u_c : 2^{\mathcal{S}} \to \mathbb{R}$ which is additively separable over students. In other words, for each college c, and for each set of students $S \subseteq \mathcal{S}$, $u_c(S) = \sum\limits_{s \in S} u_c(s)$. Let $u \equiv (u_c)_{c \in \mathcal{C}}$ be a utility profile. Each college c assigns a **value** to each student s, which we denote by $v^c_s \in \mathbb{R}_+$. Let $v^c \equiv \{v^c_{s_1}, v^c_{s_2}, \ldots, v^c_{s_m}\} \in \mathbb{R}^m_+$ be the value profile of college c. For each college, no two students have the same value, i.e. for each college c, and for each pair $s, s' \in \mathcal{S}$, $v^c_s \neq v^c_{s'}$. Welfare of a college c from admitting a student s is the value c assigns to s, i.e. $u_c(s) = v^c_s$.

Each student has quasi-linear preferences over college-stipend bundles. In other words, each student s has preference relation R_s defined over $\{\mathcal{C} \cup \{\emptyset_c\}\} \times \mathbb{R}$. Let $\mathcal{R}$ be the set of all preference relations. By $(c, x)\ P_s\ (c', x')$ we mean that student s prefers (c, x) to (c', x'). Also, by $(c, x)\ I_s\ (c', x')$ we mean that student s is indifferent between these two bundles. Let $R \equiv (R_s)_{s \in \mathcal{S}}$ be the **preference profile** of students. Let $\mathcal{R}^{\mathcal{S}}$ be the set of all preference profiles. Each student always has the option of staying at home. This option is the bundle $(\emptyset_c, 0)$. For each college c, each student s has a smallest stipend $\ell^s_c(R_s) \in \mathbb{R}_+$, which we call his **lower bound**, that he would like to receive in order to attend college c. This lower bound is derived from the preference of the student in the following way: for each student s and each college c,

$$\ell^s_c(R_s) = \begin{cases} 0 & \text{if } (c, 0)\ P_s\ (\emptyset_c, 0) \\ x & \text{if } (c, x)\ I_s\ (\emptyset_c, 0). \end{cases}$$

Let $\ell^s(R_s) \equiv \{\ell^s_{c_1}(R_s), \ell^s_{c_2}(R_s), \ldots, \ell^s_{c_m}(R_s)\} \in \mathbb{R}^m_+$ be lower bound profile of student s. A typical preference relation of a student is shown in the figure below

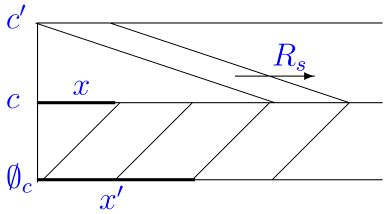


Figure 1: *Preference of student s is shown. He is indifferent between attending college c with stipend x and staying at home, $(c, x)\ I_s\ (\emptyset_c, 0)$. Therefore, $\ell^s_c(R_s) = x$. Also student s is indifferent between attending college c' with no stipend and staying at home with x' amount of money. This implies that, $(c, 0)\ P_s\ (\emptyset_c, 0)$ and therefore $\ell^s_c(R_s) = 0$.*

A **problem** is a list $\pi \equiv (q, B, m, (v_c)_{c\in\mathcal{C}}, u, R)$. Let Π be the set of all problems. Let $x^c_s \in \mathbb{R}_+$ be a **stipend** offered by college c to student s. Let $x \equiv (x_{s_1}, x_{s_2}, \ldots, x_{s_m}) \in \mathbb{R}^{\mathcal{S}}_+$ be a stipend list. An **allocation for** $\boldsymbol{\pi}$ is a **matching** $\mu : \mathcal{S} \to \mathcal{C} \cup \emptyset_c$ of students to colleges, together with a **stipend list** $x = (x^{\mu(s_1)}_{s_1}, x^{\mu(s_2)}_{s_2}, \ldots, x^{\mu(s_m)}_{s_m}) \in \mathbb{R}^{\mathcal{S}}_+$. We can also denote an allocation (μ, x) as a set of triples $\{(s, \mu(s), x^{\mu(s)}_s)_{s\in\mathcal{S}}\}$. An allocation (μ, x) is feasible if it satisfies the following requirements:

$$
\begin{aligned}
&\forall c \in \mathcal{C},\ \ |\mu^{-1}(c)| \le q_c.\\
&\forall s \in \mathcal{S},\ \ |\mu(s)| \le 1.\\
&\forall c \in \mathcal{C},\ \ \sum_{s\in\mu^{-1}(c)} x^c_s \le B^c.\\
&\forall s \in \mathcal{S},\ \ x^{\mu(s)}_s \le m^{\mu(s)}.
\end{aligned}
$$

Let $\boldsymbol{\mathcal{A}(\pi)}$ denote the set of all feasible allocations for $\boldsymbol{\pi}$. A **rule** $\varphi : \Pi \to \bigcup_{\pi\in\Pi} \mathcal{A}(\pi)$ associates with each problem an allocation for it.

3 Axioms

Next, we introduce several axioms. Let φ be a rule.

• In order to define our first requirement we need to introduce the notion of blocking. Let an allocation be selected for a problem. Suppose there is a college and a student such that the student is not matched to this college. Also suppose that the college and the student can come together and find a stipend such that the student prefers this college with this stipend to his initial allocation. Also the college will be better off by admitting this student even if it has to release some of the students initially matched to it. Then we say that this college-student pair blocks the initial allocation. Formally,

A **college-student pair (c, s) blocks allocation** $(\mu, (x_{s_1}^{\mu(s_1)}, x_{s_2}^{\mu(s_2)}, \ldots, x_{s_m}^{\mu(s_m)})) \in \mathcal{A}(\pi)$, if $\mu(s) \neq c$, and there are $\bar{S} \subseteq \mu^{-1}(c)$ and $x' \in \mathbb{R}_+$ such that

(1) $\sum_{s' \in \bar{S}} v_{s'}^c < v_s^c$,

(2) $1 + |\mu^{-1}(c) \backslash \{\bar{S}\}| \leq q_c$,

(3) $x' \leq \min\{m^c, \sum_{s' \in \bar{S}} x_{s'}^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c\}$,

(4) $(c, x')\ P_s\ (\mu(s), x_s^{\mu(s)})$

are satisfied.

Now we can introduce our requirement: An allocation is **pairwise stable**, if there is no college-student pair that blocks it. Let $\mathbf{PS(\pi)}$ be the set of all pairwise stable allocations for π.

Pairwise stability: For each $\pi \in \Pi$, $\varphi(\pi) \in PS(\pi)$.

• Next we define two efficiency requirements. The first one says that an allocation is chosen only if there is no other allocation that makes at least one college or student better off without making anyone else worse off. Formally, An allocation $(\mu, (x_{s_1}^{\mu(s_1)}, x_{s_2}^{\mu(s_2)}, \ldots, x_{s_m}^{\mu(s_m)}))$ is **Patreto-efficient** if there is no $(\mu', (\bar{x}_{s_1}^{\mu'(s_1)}, \bar{x}_{s_2}^{\mu'(s_2)}, \ldots, \bar{x}_{s_m}^{\mu'(s_m)})) \in \mathcal{A}(\pi)$ such that

for each $s \in \mathcal{S}$, we have $(\mu'(s), \bar{x}_s^{\mu'(s)})\ R_s\ (\mu(s), x_s^{\mu(s)})$,
for each $c \in \mathcal{C}$, we have $\sum_{s \in \mu^{-1}(c)} v_s^c \leq \sum_{s \in \mu'^{-1}(c)} v_s^c$,

and either
- there is $s \in \mathcal{S}$, such that $(\mu'(s), \bar{x}_s^{\mu'(s)})\ P_s\ (\mu(s), x_s^{\mu(s)})$,

or
- there is $c \in \mathcal{C}$, such that $\sum_{s \in \mu^{-1}(c)} v_s^c < \sum_{s \in \mu'^{-1}(c)} v_s^c$.

Let **PE($\boldsymbol{\pi}$)** be the set of all Pareto-efficient allocations for π.

Pareto-efficiency: For each $\pi \in \Pi$, $\varphi(\pi) \in PE(\pi)$.

• The second and weaker efficiency requirement says that an allocation is chosen only if there is no other allocation that makes every student and college better off. Formally,

An allocation $(\mu, (x_{s_1}^{\mu(s_1)}, x_{s_2}^{\mu(s_2)}, \ldots, x_{s_m}^{\mu(s_m)}))$ is **weakly Patreto-efficient** if there is no $(\mu', (\bar{x}_{s_1}^{\mu'(s_1)}, \bar{x}_{s_2}^{\mu'(s_2)}, \ldots, \bar{x}_{s_m}^{\mu'(s_m)})) \in \mathcal{A}(\pi)$ such that

- for each $s \in \mathcal{S}$, we have $(\mu'(s), \bar{x}_s^{\mu'(s)})\ P_s\ (\mu(s), x_s^{\mu(s)})$,

and
- for each $c \in \mathcal{C}$, we have $\sum_{s \in \mu^{-1}(c)} v_s^c < \sum_{s \in \mu'^{-1}(c)} v_s^c$.

Let **WPE($\boldsymbol{\pi}$)** be the set of all weak Pareto-efficient allocations for π.

Weak Pareto-efficiency: For each $\pi \in \Pi$, $\varphi(\pi) \in WPE(\pi)$.

4 Rules

Let π be a problem. Let $\succ$ be an order on the set of colleges $\mathcal{C}$. Let $\Gamma(\pi)$ be the set of all possible orders for π.

Our rule is associated with an algorithm that we define. There are two levels in the algorithm. In level 1, at each step each college defines the set of students to whom it may offer admission. At each step, each college offers admission to at most one student. Each college starts by offering admission to the student with the highest value among the students to whom college may offer admission. The stipend offered is the minimum of the money available to the college, and the maximal stipend the college can offer. Each student compares the offers he receives, if any, together with the option of staying at home, and tentatively accepts the one he prefers. If the student is indifferent between offers, we use a predetermined order on colleges to break ties. Next, each college defines the set of students to whom it may offer admission. Each college with at least one empty seat, offers admission to the student with the highest value among the students to whom college may offer admission. The stipend it offers is the minimum of the money that the college is left with after previous offers, and the maximal stipend the college can offer. Level 1 continues in this way until either there are no students left to whom a college can offer admission, or until all colleges are full.

In level 2, we consider the students who are not matched to any college in level 1. We design a procedure that matches these students to colleges if it is possible.

Level 1 of our algorithm is similar to Gale-Shapley's DA algorithm. What differs is that we make some adjustments in level 1. We adjust the set of students a college tentatively admits and the set of students who reject the offer of the college. Adjustments are made when a student who was tentatively accepting the offer of a college, rejects it at some later step. Once the student rejects the offer, all students who have lower value than this rejecting student for that college become available to receive an offer from this college. In other words, independent of whether those students were previously rejecting the offer of this college or not, this college will be able to offer admission to these students one more time. The algorithm is defined formally below.

Best Comes First rule, BCF

Let $\pi \in \Pi$ be a problem. Let $\succ \in \Gamma(\pi)$ be an order.
Level 1:
At each step, each college can offer admission to at most one new student. Each offer that was tentatively accepted in previous steps is still in effect. At each step, each student has the option of staying at home, that is, choosing bundle $(\emptyset_c, 0)$.

Step 0: Let M_0^c be set of students who are tentatively admitted to c. Since there is no prior step, $M_0^c \equiv \emptyset$.

Step 1: Let $O_1^c \equiv \mathcal{S}$. Each $c \in \mathcal{C}$ can offer admission only to a student in O_1^c.
Each $c \in \mathcal{C}$ with $|M_0^c| < q_c$, offers admission to $s \equiv \arg\max_{s' \in O_1^c} v_{s'}^c$, with stipend $x_{s,1}^c = \min\{m^c, B^c\}$. Each $c \in \mathcal{C}$ with $|M_0^c| = q_c$, does not offer admission to anyone.

Each $s \in \mathcal{S}$ compares the offers he receives at this step, if any, together with $(\emptyset_c, 0)$. He tentatively accepts the one he prefers and rejects the others. Students maybe indifferent between offers. To solve this issue, we use $\succ$ as a tie-breaker. We break ties in the following way:

if $(c, x_s^c)\ I_s\ (c', x_s^{c'})$, then $(c, x_s^c)\ P_s\ (c', x_s^{c'})$ if and only if $c \succ c'$,

and

if $(c, x_s^c)\ I_s\ (\emptyset_c, 0)$, then $(c, x_s^c)\ P_s\ (\emptyset_c, 0)$.

Let M_1^c be the set of students who are tentatively admitted to college c and R_1^c be the set of students who reject the offer of college c.

For each $c \in \mathcal{C}$, we define $O_2^c \equiv O_1^c \setminus \{R_1^c \cup M_1^c\}$ and proceed to Step 2.

Step $t = 2, 3, \ldots$: Let $O_t^c \equiv O_{t-1}^c \setminus \{R_{t-1}^c \cup M_{t-1}^c\}$. Each $c \in \mathcal{C}$ can offer admission only to a student in O_t^c.

Each $c \in \mathcal{C}$ with $|M_{t-1}^c| < q_c$, offers admission to $s \equiv \arg\max_{s' \in O_t^c} v_{s'}^c$, with stipend $x_{s,t}^c = \min\{m^c, B^c - \sum_{s' \in M_{t-1}^c} x_{s',t-1}^c\}$. Each $c \in \mathcal{C}$ with $|M_0^c| = q_c$, does not offer admission to anyone.

Each $s \in \mathcal{S}$ compares the offers he receives at this step, if any, the offer that he tentatively accepted at previous steps, if any, together with $(\emptyset_c, 0)$. He tentatively accepts the one he prefers and rejects the others. When student is indifferent, we use $\succ$ to break ties as before. Each student who tentatively accepts the offer of c joins the set M_t^c and each student who rejects the offer of c joins the set R_t^c.

For each $c \in \mathcal{C}$, we define $O_{t+1}^c = O_t^c \setminus \{R_t^c \cup M_t^c\}$ and proceed to Step t+1.

Adjustments made at each step in Level 1:
At each step t, each $s \in \mathcal{S}$ joins R_t^c

(1) if s receives an offer from c at step t but rejects it.

(2) if s tentatively accepted the offer of c at previous steps but receives a better offer at step t, and rejects the offer of c.

In case (2), we revise the sets M_t^c and R_t^c as follows:

- Let $D_t^c \equiv \{s' \in \mathcal{S} | s' \in M_{t-1}^c \text{ and } s' \in R_t^c\}$. Let $s \equiv \arg\max_{s' \in D_t^c} v_{s'}^c$. Then $M_t^c \equiv \{s' \in M_{t-1}^c, \text{ s.t. } v_{s'}^c > v_s^c\}$. [In other words, c gives up all the tentatively admitted students who have lower value than s for c. (s is the one with the highest value for c among the students who were tentatively admitted to c and reject it at step t.) These students will be available to receive an offer from c at next step, that is, they join O_{t+1}^c.]

- Let $D_t^c \equiv \{s' \in \mathcal{S} | s' \in M_{t-1}^c \text{ and } s' \notin M_t^c\}$. Let $s \equiv \arg\max_{s' \in D_t^c} v_{s'}^c$, then $R_t^c \equiv \{s' \in R_{t-1}^c, \text{ s.t. } v_{s'}^c \geq v_s^c\} \cup \{s\}$. [In other words, students who reject the offer of c at previous steps and have lower value than s for c (s is the one with the highest value for c among the students who were tentatively admitted to c and reject it at step t.) will be available to receive an offer from c at next step, that is, they join O_{t+1}^c.]

We call these adjustments as *restart*.

For each $c \in \mathcal{C}$, let $O_t^c \equiv O_{t-1}^c \setminus \{R_t^c \cup M_t^c\}$. At step t, c can offer admission only to a student in O_t^c. This set includes:

(a) All the students with lower value than s for c, where s is as in case (2) above. [i.e. he tentatively accepted an offer before, but rejected it in this period.]

(b) All the students to whom college have not offered admission yet.

The first level of the algorithm ends at step $\bar{t}$ at which, each $c \in \mathcal{C}$ either

- is full, i.e. $|M_{\bar{t}}^c| = q_c$,

or

- has empty seats, i.e. $|M_{\bar{t}}^c| < q_c$ but there is no student it can offer admission to, i.e. $O_{\bar{t}+1}^c = \emptyset$.

For each $c \in \mathcal{C}$, matching at the end of level 1 is $\mu_1^{-1}(c) = M_{\bar{t}}^c$.

Level 2:

Step 0:

For each $c \in \mathcal{C}$, let M_0^c be the set of the students who are matched to c at the beginning of level 2. Since prior to level 2 there are matches resulted from level 1, then for each $c \in \mathcal{C}$, we have $M_0^c \equiv \mu_1^{-1}(c)$. Also, for each $c \in \mathcal{C}$, let $M_0^c(s)$ be the set of sets of students in M_0^c who have lower value than s for c. Formally,

$$M_0^c(s) \equiv \{\bar{S} \subseteq M_0^c \text{ s.t. } \sum_{s' \in \bar{S}} v_{s'}^c < v_s^c\}.$$

Step 1:

Let U_1 be the set of the students who are unmatched at the beginning of step 1. All students who are not matched at level 1 are unmatched at step 1,

$$U_1 \equiv \mathcal{S} \setminus \{\bigcup_{c \in \mathcal{C}} M_0^c\} \equiv \mathcal{S} \setminus \{\bigcup_{c \in \mathcal{C}} \mu_1^{-1}(c)\}.$$

College **c can offer admission to student s** if

(i) Either $|M_0^c| < q_c$ or $M_0^c(s) \neq \emptyset$

and

(ii) $\bar{x}_s^c \geq \ell_c^s(R_s)$

where

$$\bar{x}_s^c \equiv \min\{m^c, \max_{\bar{S}\in M_0^c(s)}\{\sum_{s'\in\bar{S}} x_{s'}^c\} + B^c - \sum_{s''\in M_0^c} x_{s''}\}.$$

Let $\boldsymbol{C_1(s)}$ be the set of colleges that can offer admission to s at step 1. For each $s \in U_1$, determine $C_1(s)$. If for each $c \in \{c' \in C_1(s) | \text{for each } c'' \in C_1(s)\backslash c', \ (c', \bar{x}_s^{c'}) \ R_s \ (c'', \bar{x}_s^{c''})\}$, we have $\bar{c} \succ c$, then college $\bar{c} \in C_1(s)$ **wins the right to offer admission to $\boldsymbol{s}$**.

Offers are made in the following way: For each $c \in \mathcal{C}$, let O_1^c to be the set of students in U_1 to whom c wins the right to offer admission. Rank the students in O_1^c in increasing order of values. Start from the student with lowest value, call him s_1. There are two possibilities

Case 1: The set $C_1(s_1)$ is singleton. Then,

– find $\bar{S} \in M_0^c(s)$ with $\sum_{s'\in\bar{S}} x_{s'} + B^c - \sum_{s''\in M_0^c} x_{s''}^c \geq \ell_c^{s_1}(R_{s_1})$ that minimizes $\sum_{s'\in\bar{S}} v_s^c$. [Note that $\bar{S}$ can be a singleton]

Then, c offers admission to s_1 with stipend $x_{s_1}^c \equiv \min\{m^c, \sum_{s'\in\bar{S}} x_{s'}^c + B^c - \sum_{s''\in M_0^c} x_{s''}^c\}$ and releases all the students in $\bar{S}$.

Case 2: The set $C_1(s_1)$ is not singleton. Then, there is $\widehat{c} \in C_1(s_1)\backslash\{c\}$, such that for each $\bar{c} \in C_1(s_1)\backslash\{c, \widehat{c}\}$, we have $(\widehat{c}, \bar{x}_{s_1}^{\widehat{c}}) \ R_{s_1} \ (\bar{c}, \bar{x}_{s_1}^c)$. Next,

– find minimal x, with $x \geq \ell_c^{s_1}(R_{s_1})$ satisfying $(c, x) \ R_{s_1} \ (\widehat{c}, \bar{x}_i^{\widehat{c}})$

– find $\bar{S} \in M_0^c(s)$ with $\sum_{s'\in\bar{S}} x_{s'} + B^c - \sum_{s''\in M_0^c} x_{s''}^c \geq x$ that minimizes $\sum_{s'\in\bar{S}} v_s^c$. [Note that $\bar{S}$ can be a singleton]

Then, c offers admission to s_1 with stipend $x_{s_1}^c \equiv \min\{m^c, \sum_{s'\in\bar{S}} x_{s'}^c + B^c - \sum_{s''\in M_0^c} x_{s''}^c\}$ and releases all the students in $\bar{S}$.

After admitting s_1, we find the next student with lowest value in O_1^c, call him s_2. We repeat the procedure for s_2. We proceed in the similar way and repeat the procedure for all the students in O_1^c. Step 1 finishes when each $c \in \mathcal{C}$ makes offers to all students in O_1^c.

Set of unmatched students at the beginning of step 2, U_2, includes

- students in U_1 to whom no college made an offer.
- students that were released from the colleges at step 1.

We define U_2 and move to step 2.

Level 2 continues until step t^*, at which either for each $s \in U_{t^*+1}$, there is no college that can offer admission to him or $U_{t^*+1} = \emptyset$. For each $c \in \mathcal{C}$ the resulting matching is $\mu^{-1}(c) = M_{t^*}^c$.

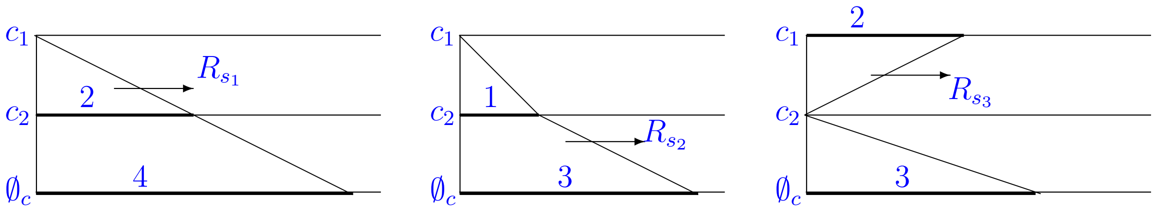
When level 2 ends, we adjust the stipends. For each $c \in \mathcal{C}$, let $\mu_2^{-1}(c)$ be the set of students who were admitted in level 2. If there is some money left at the end of the algorithm, that is, if $B^c - \sum_{s \in \mu^{-1}(c)} x_s^c > 0$, then this amount is allocated to the students in $\mu_2^{-1}(c)$ in the following way: First, we find the student with the highest value in $\mu_2^{-1}(c)$ for c and adjust his stipend either up to m^c, or by the money left in hand. Then, if there is still some money left, we find the student with the second highest value in $\mu_2^{-1}(c)$ for c, and adjust his stipend in the same way. We adjust the stipends of all the students in $\mu_2^{-1}(c)$ in the same way.

The Best Comes First Algorithm is well-defined and terminates in finite steps. Level 1 terminates because there are finite number of students and therefore each college can make finite number of offers. Although colleges can re-offer to some of the students at the later steps, the number of the students that they can offer decreases every time they restart. Level 2 terminates in finite number of steps because at every step there is at least one college that becomes better off and no college becomes worse off. Since the total welfare that a college can get is bounded by the sum of the values of the students for that college, then level 2 terminates at some step.

Next, we give several examples to show how the algorithm works.

Example 1: Let $\pi \in \Pi$. Let $\mathcal{C} = \{c_1, c_2\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$. Let $c_1 \succ c_2$. Let $v_{c_1} = (6, 5, 4)$, $v_{c_2} = (3, 5, 4)$, $B = (10, 6)$, $m = (7, 6)$ and $q = (2, 1)$.

Preferences of students are as follows



Therefore, $\ell^{s_1}(R_{s_1}) = (0, 0)$, $\ell^{s_2}(R_{s_2}) = (0, 0)$, and $\ell^{s_3}(R_{s_3}) = (0, 0)$.

Level 1:

Step 1:

We have $O_1^{c_1} = \{\boldsymbol{s_1}, s_2, s_3\}$ and $O_1^{c_2} = \{s_1, \boldsymbol{s_2}, s_3\}$. College c_1 offers admission to s_1 with stipend $x_{s_1}^{c_1} = \min\{7, 10\} = 7$, and c_2 offers admission to s_2 with stipend $x_{s_2}^{c_2} = \min\{7, 6\} = 6$. Both s_1 and s_2 tentatively accept the offers. At the end of step 1, we have $M_1^{c_1} = \{s_1\}$, $R_1^{c_1} = \emptyset$, $M_1^{c_2} = \{s_2\}$, and $R_1^{c_2} = \emptyset$.

Step 2:

We have $O_2^{c_1} = \{\boldsymbol{s_2}, s_3\}$ and $O_2^{c_1} = \{s_1, s_3\}$. College c_1 offers admission to s_2 with stipend $x_{s_2}^{c_1} = \min\{7, 3\} = 3$, and since $|M_1(c_2)| = q_{c_2}$, college c_2 does not offer admission to anyone. Student s_2 compares offers and since $(c_2, 6)\ P_{s_2}\ (c_1, 3)$, he tentatively accepts the offer of c_2. At the end of step 2 we have $M_2^{c_1} = \{s_1\}$, $R_2^{c_1} = \{s_2\}$, $M_2^{c_2} = \{s_2\}$, and $R_2^{c_2} = \emptyset$.

Step 3:

We have $O_3^{c_1} = \{\boldsymbol{s_3}\}$ and $O_3^{c_1} = \{s_1, s_3\}$. College c_1 offers admission to s_3 with stipend $x_{s_3}^{c_1} = \min\{7, 3\} = 3$, and since $|M_1(c_2)| = q_{c_2}$, college c_2 does not offer admission to anyone. Student s_3 tentatively accepts the offer of c_1. At the end of step 3 we have $M_3^{c_1} = \{s_1, s_3\}$, $R_2^{c_1} = \{s_2\}$, $M_2^{c_2} = \{s_2\}$, and $R_2^{c_2} = \emptyset$.

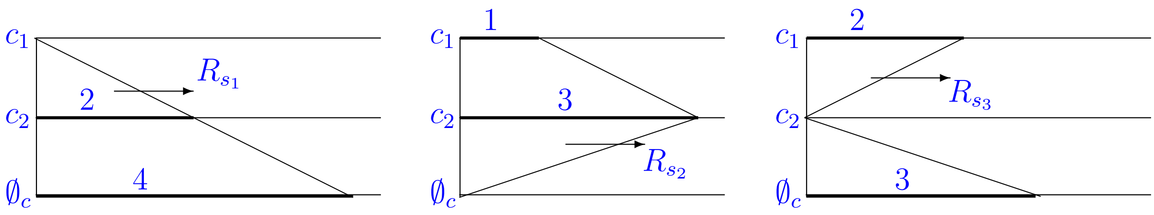
Since all colleges are full, algorithm stops. The final allocation is

$BCF(\pi) = \{(s_1, c_1, 7), (s_2, c_2, 6), (s_3, c_1, 3)\}$.

In our next example, there is a case when staying at home is preferred to attending some college with certain stipend.

Example 2: Let $\pi \in \Pi$. Let $\mathcal{C} = \{c_1, c_2\}$ and $\mathcal{S} = \{s_1, s_2, s_3\}$. Let $c_1 \succ c_2$. Let $v_{c_1} = (6, 5, 4)$, $v_{c_2} = (3, 5, 4)$, $B = (9, 2)$, $m = (5, 2)$ and $q = (2, 1)$.

Preferences of students are as follows



Therefore, $\ell^{s_1}(R_{s_1}) = (0, 0)$, $\ell^{s_2}(R_{s_2}) = (1, 3)$, and $\ell^{s_3}(R_{s_3}) = (0, 0)$.

Level 1:

Step 1:

We have $O_1^{c_1} = \{\boldsymbol{s_1}, s_2, s_3\}$ and $O_1^{c_2} = \{s_1, \boldsymbol{s_2}, s_3\}$. College c_1 offers admission to s_1 with stipend $x_{s_1}^{c_1} = \min\{5, 10\} = 5$, and c_2 offers admission to s_2 with stipend $x_{s_2}^{c_2} = \min\{2, 2\} = 2$. Student s_1 tentatively accept the offer of c_1. Since $(\emptyset_c, 0)\ P_{s_2}\ (c_2, 2)$, student s_2 rejects the offer of c_2 and chooses $(\emptyset_c, 0)$. At the end of step 1 we have $M_1^{c_1} = \{s_1\}$, $R_1^{c_1} = \emptyset$, $M_1^{c_2} = \emptyset$, and $R_1^{c_2} = \{s_2\}$.

Step 2:

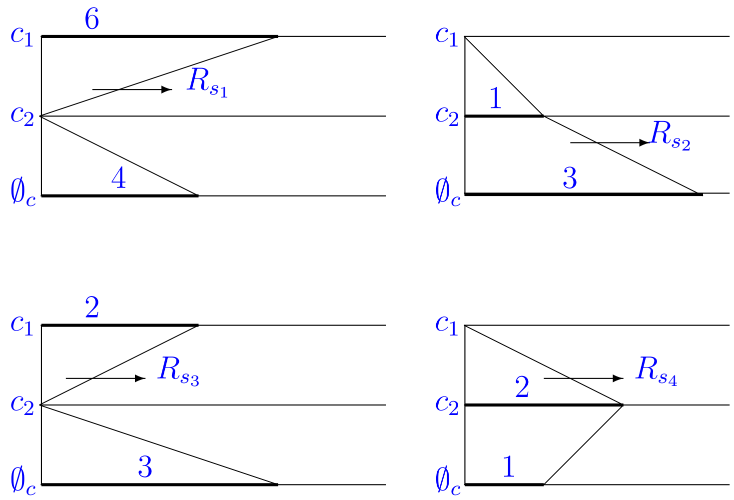
We have $O_2^{c_1} = \{\boldsymbol{s_2}, s_3\}$ and $O_2^{c_1} = \{s_1, \boldsymbol{s_3}\}$. College c_1 offers admission to s_2 with stipend $x_{s_2}^{c_1} = \min\{5, 4\} = 4$, and c_2 offers admission to s_3 with stipend $x_{s_3}^{c_2} = \min\{2, 2\} = 2$. Both s_2 and s_3 tentatively accept the offers. At the end of step 2 we have $M_2^{c_1} = \{s_1, s_2\}$, $R_2^{c_1} = \emptyset$, $M_2^{c_2} = \{s_3\}$, and $R_2^{c_2} = \{s_2\}$.

Since all colleges are full, algorithm stops. The final allocation is

$BCF(\pi) = \{(s_1, c_1, 5), (s_2, c_1, 4), (s_3, c_2, 2)\}$.

Next, we consider example in which we have adjustments during the algorithm.

Example 3: Let $\pi \in \Pi$. Let $\mathcal{C} = \{c_1, c_2\}$ and $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$. Let $c_2 \succ c_1$. Let $v_{c_1} = (6, 5, 3, 4)$, $v_{c_2} = (4, 5, 6, 3)$, $B = (6, 8)$, $m = (6, 6)$ and $q = (2, 3)$.
Preferences of students are as follows



Therefore, $\ell^{s_1}(R_{s_1}) = (2, 0)$, $\ell^{s_2}(R_{s_2}) = (0, 0)$, $\ell^{s_3}(R_{s_3}) = (0, 0)$, and $\ell^{s_4}(R_{s_4}) = (0, 1)$.

Level 1:

Step 1:

We have $O_1^{c_1} = \{\boldsymbol{s_1}, s_2, s_3, s_4\}$ and $O_1^{c_2} = \{s_1, s_2, \boldsymbol{s_3}, s_4\}$. College c_1 offers admission to s_1 with stipend $x_{s_1}^{c_1} = \min\{6, 6\} = 6$, and c_2 offers admission to s_3 with stipend $x_{s_3}^{c_2} = \min\{6, 8\} = 6$. Both s_1 and s_3 tentatively accept the offers. At the end of step 1 we have $M_1^{c_1} = \{s_1\}$, $R_1^{c_1} = \emptyset$, $M_1^{c_2} = \{s_3\}$, and $R_1^{c_2} = \emptyset$.

Step 2:

We have $O_2^{c_1} = \{\boldsymbol{s_2}, s_3, s_4\}$ and $O_2^{c_2} = \{s_1, \boldsymbol{s_2}, s_4\}$. College c_1 offers admission to s_2 with stipend $x_{s_2}^{c_1} = \min\{6, 0\} = 0$, and c_2 offers admission to s_2 with stipend $x_{s_2}^{c_2} = \min\{6, 2\} = 2$. Student s_2 compares the offers and since $(c_2, 2)\ P_{s_2}\ (c_1, 0)$, he tentatively accepts the offer of c_2. At the end of step 2 we have $M_2^{c_1} = \{s_1\}$, $R_2^{c_1} = \{s_2\}$, $M_2^{c_2} = \{s_2, s_3\}$, $R_2^{c_2} = \emptyset$.

Step 3:

We have $O_3^{c_1} = \{s_3, \boldsymbol{s_4}\}$ and $O_3^{c_2} = \{\boldsymbol{s_1}, s_4\}$. College c_1 offers admission to s_4 with stipend $x_{s_4}^{c_1} = \min\{6, 0\} = 0$, and c_2 offers admission to s_1 with stipend $x_{s_1}^{c_2} = \min\{6, 0\} = 0$. Student s_1 compares offers and is indifferent between them, that is $(c_2, 0)\ I_{s_1}\ (c_1, 6)$. Therefore we use $\succ$ to break the tie. Since $c_2 \succ c_1$, student s_1 tentatively accepts the offer of c_2. Student s_4 tentatively accepts the offer of c_1. At the end of step 3 we have $M_3^{c_1} = \emptyset$, $R_3^{c_1} = \{s_1\}$, $M_3^{c_2} = \{s_1, s_2, s_3\}$, $R_3^{c_2} = \emptyset$. At the end of this step we adjusted our sets $M_3^{c_1}$ and $R_3^{c_1}$ because a previously accepting student, rejected the offer at this step.

Step 4:

We have $O_4^{c_1} = \{\boldsymbol{s_2}, s_3, s_4\}$ and $O_4^{c_2} = \{s_4\}$. College c_1 offers admission to s_2 with stipend $x_{s_2}^{c_1} = \min\{6, 6\} = 6$, and since $|M_3(c_2)| = q_{c_2}$, college c_2 does not offer admission to anyone. Student s_2 compares the offers and since $(c_1, 6)\ P_{s_2}\ (c_2, 2)$, he tentatively accepts the offer of college c_1. At the end of step 4 we have $M_4^{c_1} = \{s_2\}$, $R_4^{c_1} = \{s_1\}$, $M_4^{c_2} = \{s_3\}$, $R_4^{c_2} = \{s_2\}$. At the end of this step we adjusted our sets $M_4^{c_2}$ and $R_4^{c_2}$ because a previously accepting student, rejected the offer at this step.

Step 5:

We have $O_5^{c_1} = \{s_3, \boldsymbol{s_4}\}$ and $O_5^{c_2} = \{\boldsymbol{s_1}, s_4\}$. College c_1 offers admission to s_4 with stipend $x_{s_4}^{c_1} = \min\{6, 0\} = 1$, and c_2 offers admission to s_1 with stipend $x_{s_1}^{c_2} = \min\{6, 2\} = 2$. Both s_1 and s_4 tentatively accept the offers. At the end of step 5 we have $M_5^{c_1} = \{s_2, s_4\}$, $R_5^{c_1} = \{s_1\}$, $M_5^{c_2} = \{s_1, s_3\}$, $R_5^{c_2} = \{s_2\}$.

Step 6:

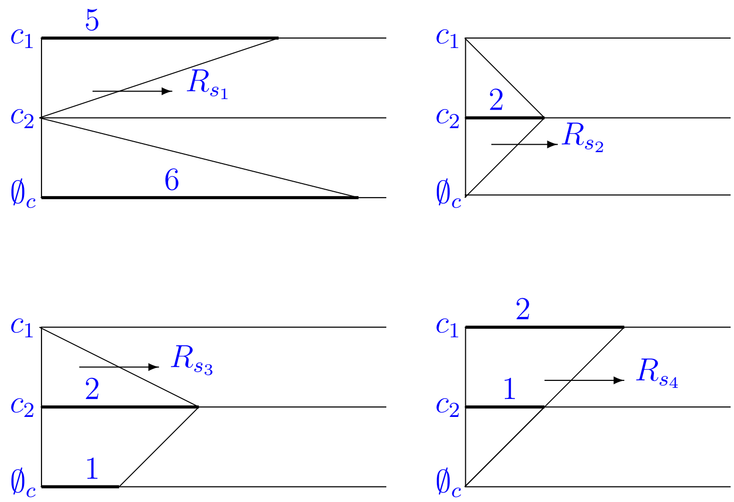
We have $O_6^{c_1} = \{s_3\}$ and $O_6^{c_2} = \{\boldsymbol{s_4}\}$. College c_2 offers admission to s_4 with stipend $x_{s_4}^{c_2} = \min\{6, 0\} = 0$, and since $|M_5(c_1)| = q_{c_1}$, college c_1 does not offer admission to anyone. Student s_4 compares the offers and since $(c_1, 0)\ P_{s_4}\ (c_2, 0)$, he tentatively accepts the offer of c_1. At the end of step 6 we have $M_6^{c_1} = \{s_2, s_4\}$, $R_6^{c_1} = \{s_1\}$, $M_6^{c_2} = \{s_1, s_3\}$, $R_6^{c_2} = \{s_2, s_4\}$.

Since colleges are either full or have no student to offer admission to algorithm stops. The final allocation is $BCF(\pi) = \{(s_1, c_2, 2), (s_2, c_1, 6), (s_3, c_2, 6), (s_4, c_1, 0)\}$.

Our last example considers the case when we have second level of the algorithm.

Example 4: Let $\pi \in \Pi$. Let $\mathcal{C} = \{c_1, c_2\}$ and $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$. Let $c_2 \succ c_1$. Let $v_{c_1} = (6, 5, 4, 3)$ and $v_{c_2} = (3, 5, 4, 2)$. Let $B = (7, 7)$, $m = (5, 6)$ and $q = (1, 3)$.

Preferences of students are as follows



Therefore, $\ell^{s_1}(R_{s_1}) = (0, 0)$, $\ell^{s_2}(R_{s_2}) = (1, 0)$, $\ell^{s_3}(R_{s_3}) = (0, 1)$ and $\ell^{s_4}(R_{s_4}) = (2, 1)$.

Level 1:

Step 1:

We have $O_1^{c_1} = \{\boldsymbol{s_1}, s_2, s_3, s_4\}$ and $O_1^{c_2} = \{s_1, \boldsymbol{s_2}, s_3, s_4\}$. College c_1 offers admission to s_1 with stipend $x_{s_1}^{c_1} = \min\{5, 7\} = 5$, and c_2 offers admission to s_2 with stipend $x_{s_2}^{c_2} = \min\{6, 7\} = 6$. Both s_1 and s_2 tentatively accept the offers. At the end of step 1, we have $M_1^{c_1} = \{s_1\}$, $R_1^{c_1} = \emptyset$, $M_1^{c_2} = \{s_2\}$, and $R_1^{c_2} = \emptyset$.

Step 2:

We have $O_2^{c_1} = \{s_2, s_3, s_4\}$ and $O_2^{c_2} = \{s_1, \boldsymbol{s_3}, s_4\}$. College c_2 offers admission to s_3 with stipend $x_{s_3}^{c_2} = \min\{6, 1\} = 1$ and since $|M_1^{c_1}| = 1$, college c_1 does not offer admission to anyone. Student s_3 tentatively accepts the offer of c_2. At the end of step 2, we have $M_2^{c_1} = \{s_1\}$, $R_2^{c_1} = \emptyset$, $M_2^{c_2} = \{s_2, s_3\}$, and $R_2^{c_2} = \emptyset$.

Step 3:

We have $O_3^{c_1} = \{s_2, s_3, s_4\}$ and $O_3^{c_2} = \{\boldsymbol{s_1}, s_4\}$. College c_2 offers admission to s_1 with stipend $x_{s_1}^{c_2} = \min\{6, 0\} = 0$ and since $|M_1^{c_1}| = 1$, college c_1 does not offer admission to anyone. Student s_1 compares offers and is indifferent between them, that is $(c_2, 0)\ I_{s_1}\ (c_1, 5)$. Therefore we use $\succ$ to break the tie. Since $c_2 \succ c_1$, student s_1 tentatively accepts the offer of c_2. At the end of step 3, we have $M_3^{c_1} = \emptyset$, $R_3^{c_1} = \{s_1\}$, $M_3^{c_2} = \{s_1, s_2, s_3\}$, and $R_3^{c_2} = \emptyset$.

Step 4:

We have $O_4^{c_1} = \{\boldsymbol{s_2}, s_3, s_4\}$ and $O_4^{c_2} = \{s_4\}$. College c_1 offers admission to s_2 with stipend $x_{s_2}^{c_1} = \min\{5, 7\} = 5$ and since $|M_3^{c_1}| = 3$, college c_2 does not offer admission to anyone. Student s_2 compares offers and since $(c_1, 5)\ P_{s_2}\ (c_2, 6)$, he tentatively accepts the offer of c_1. At the end of step 4, we have $M_4^{c_1} = \{s_2\}$, $R_4^{c_1} = \{s_1\}$, $M_4^{c_2} = \emptyset$, and $R_4^{c_2} = \{s_2\}$. At the end of this step we adjusted our sets $M_4(c_2)$ because a previously accepting student, rejected the offer at this step.

Step 5:

We have $O_5^{c_1} = \{s_3, s_4\}$ and $O_5^{c_2} = \{s_1, \boldsymbol{s_3}, s_4\}$. College c_2 offers admission to s_3 with stipend $x_{s_3}^{c_2} = \min\{6, 7\} = 4$, and since $|M_4^{c_1}| = 1$, college c_1 does not offer admission to anyone. Students s_3 tentatively accepts the offer of c_2. At the end of step 5, we have $M_5^{c_1} = \{s_2\}$, $R_5^{c_1} = \{s_1\}$, $M_5^{c_2} = \{s_3\}$, and $R_5^{c_2} = \{s_2\}$.

Step 6:

We have $O_6^{c_1} = \{s_3, s_4\}$ and $O_6^{c_2} = \{\boldsymbol{s_1}, s_4\}$. College c_2 offers admission to s_1 with stipend $x_{s_1}^{c_2} = \min\{6, 1\} = 1$, and since $|M_5^{c_1}| = 1$, college c_1 does not offer admission to anyone. Students s_1 tentatively accepts the offer of c_2. At the end of step 6, we have $M_6^{c_1} = \{s_2\}$, $R_6^{c_1} = \{s_1\}$, $M_6^{c_2} = \{s_1, s_3\}$, and $R_6^{c_2} = \{s_2\}$.

Step 7:

We have $O_7^{c_1} = \{s_3, s_4\}$ and $O_7^{c_2} = \{\boldsymbol{s_4}\}$. College c_2 offers admission to s_4 with stipend $x_{s_4}^{c_2} = \min\{6, 0\} = 0$, and since $|M_6^{c_1}| = 1$, college c_1 does not offer admission to anyone. Since $(\emptyset, 0)\ P_{s_4}\ (c_2, 0)$, students s_4 rejects the offer of college c_2. At the end of step 7, we have $M_7^{c_1} = \{s_2\}$, $R_7^{c_1} = \{s_1\}$, $M_7^{c_2} = \{s_1, s_3\}$, and $R_7^{c_2} = \{s_2, s_4\}$.

Level 1 of algorithm is finishes since $|M_7^{c_1}| = 1$ and $O_8^{c_2} = \emptyset$.

Level 2: We have $U_1 = \{s_4\}$.
Step 1: Since $v_{s_2}^{c_1} > v_{s_4}^{c_1}$ and $q_{c_1} = 1$, college c_1 can not offer admission to student s_4. For c_2, we have $v_{s_3}^{c_2} > v_{s_4}^{c_2}$ and $v_{s_1}^{c_2} > v_{s_4}^{c_2}$. But c_2 has empty seat, $|M_1^{c_2}| < q_{c_2} = 3$. The stipend c_2 can offer to student s_4 is $x_{s_4}^{c_2} = \min\{B^{c_2} - x_{s_3}^{c_2} - x_{s_1}^{c_2}, 2\} = 0$. But since $x_{s_4}^{c_2} = 0 < 2 = \ell_{c_2}^{s_4}$, college c_2 can not offer admission to student s_4.

Thus level 2 terminates and allocation is $BCF(\pi) = \{(s_1, c_2, 1), (s_2, c_1, 6), (s_3, c_2, 6), s_4\}$.

5 Results

Theorem 1 : *The pairwise stable set is non-empty. Best Comes First Algorithm produces an outcome that is pairwise stable.*

Proof. Let $(\mu, (x_{s_1}^{\mu(s_1)}, x_{s_2}^{\mu(s_2)}, \ldots, x_{s_m}^{\mu(s_m)}))$ be an allocation selected by BCF algorithm. Suppose by contradiction that there is a college-student pair (c, s) such that $\mu(s) \neq c$, and there are $\bar{S} \subseteq \mu^{-1}(c)$ and $x' \in \mathbb{R}_+$ such that

(1) $\sum_{s' \in \bar{S}} v_{s'}^c < v_s^c$,

(2) $1 + |\mu^{-1}(c) \backslash \{\bar{S}\}| \leq q_c$,

(3) $x' \leq \min\{m^c, \sum_{s' \in \bar{S}} x_{s'}^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c\}$,

(4) $(c, x')\ P_s\ (\mu(s), x_s^{\mu(s)})$

are satisfied.

We consider several cases:
Case 1: Let $s \notin \mu_1^{-1}(c)$ and $s \notin \mu^{-1}(c)$

Subcase 1-1: There is no $s' \in \mu^{-1}(c)$ such that $v_{s'}^c > v_s^c$.

Sub-subcase 1-1-1: There is $\bar{c} \in \mathcal{C} \backslash \{c\}$ such that $s \in \mu_1^{-1}(\bar{c})$.

(1) Suppose $\bar{c}$ does not release s in level 2. Then s receives an offer from c with stipend $x_s^c = \min\{m^c, B^c\}$ before any student $s'' \in \mu_1^{-1}(c)$ with $v_{s''}^c < v_s^c$. But since s rejects this offer, it is not possible that

$$(c, x)\ P_s\ (\mu(s), x_s^{\mu(s)}) \ \text{ where}$$
$$x \leq \min\{m^c, \sum_{s' \in \bar{S}} x_{s'}^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c\} \leq \min\{m^c, B^c\}$$

But this contradicts our assumption that s prefers c with stipend x to his allocation at μ.

(2) Suppose $\bar{c}$ releases s at some step t in level 2, i.e. $s \in U_t$ for some t. Every college in $C_t(s)$ can offer admission to s when he is in U_t and only college with the best offer (according to R_s) wins the right to offer to s. But then $s \notin \mu^{-1}(c)$ implies either of two possibilities;

(i) There is $\widehat{c} \in \mathcal{C} \backslash \{c\}$ such that either $(\widehat{c}, x_s^{\widehat{c}})\ P_s\ (c, \min\{m^c, \max_{\hat{S} \in M_t^c(s)} (\sum_{\bar{s} \in \hat{S}} x_{\bar{s}}^c + B^c - \sum_{k \in \mu^{-1}(c)} x_k^c)\})$, or $(\widehat{c}, x_s^{\widehat{c}})\ I_s\ (c, \min\{m^c, \max_{\hat{S} \in M_t^c(s)} (\sum_{\bar{s} \in \hat{S}} x_{\bar{s}}^c + B^c - \sum_{k \in \mu^{-1}(c)} x_k^c)\})$ and $\hat{c} \succ c$. [In other words, c couldn't win the right to offer admission to s, and some other college $\widehat{c}$ offered admission to him], or

(ii) Maximum stipend that c can offer to s is $\bar{x}_s^c = \min\{m^c, \max_{\hat{S} \in M_t^c(s)} (\sum_{\bar{s} \in \hat{S}} x_{\bar{s}}^c + B^c - \sum_{k \in \mu^{-1}(c)} x_k^c)\} < \ell_c^s(R_s)$. But then by our claim we get

$$(c, x)\ R_s\ (\mu(s), x_s^{\mu(s)})\ R_s\ (\emptyset_c, 0)\ P_s\ (c, \bar{x}_s^c), \text{ where}$$

$$x \leq \min\{m^c, \sum_{s' \in \bar{S}} x_{s'}^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c\} \leq \min\{m^c, \max_{\hat{S} \in M_t^c(s)} (\sum_{\bar{s} \in \hat{S}} x_{\bar{s}}^c + B^c - \sum_{k \in \mu^{-1}(c)} x_k^c)\} = \bar{x}_s^c.$$

But this contradicts our assumption that s prefers c with stipend x to his allocation at μ. Therefore it is not possible that s prefers c with stipend x to his allocation at μ. Even if the college which wins the right to offer admission to s releases him at some later step same intuition applies. Since $s \notin \mu_c^{-1}$, then c couldn't win the right to offer admission to s again.

Sub-subcase 1-1-2: There is no $\bar{c} \in \mathcal{C}$ such that $s \in \mu_1^{-1}(\bar{c})$, that is, $s \in U_1$. By similar reasoning as in part (2) in Sub-subcase 1-1-1 above it is not possible that

$$(c,x)\ P_s\ (\mu(s), x_s^{\mu(s)}),\ \text{where}$$

$$x \leq \min\{m^c, \sum_{s' \in \bar{S}} x_s^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c\} \leq \min\{m^c, \max_{\hat{S} \in M_t^c(s)} (\sum_{\bar{s} \in \hat{S}} x_{\bar{s}}^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c)\}.$$

This contradicts our assumption that s prefers c with stipend x to his match at μ.

Subcase 1-2: There is $s' \in \mu^{-1}(c)$ such that $v_{s'}^c > v_s^c$.

Sub-subcase 1-2-1: There is $s' \in \mu_1^{-1}(c)$ such that $v_{s'}^c > v_s^c$.

Let $\mu_1^{-1}(c)(s) \equiv \{s' \in \mu_1^{-1}(c)$ such that $v_{s'}^c > v_s^c\}$. In level 1, student s received offer after all the students in $\mu_1^{-1}(c)(s)$. Therefore, the stipend offered to student s was

$$x_s^c = \min\{m^c, B^c - \sum_{s' \in \mu_1^{-1}(c)(s)} x_{s'}^c\}.$$

(1) There is $\bar{c} \in \mathcal{C} \backslash c$ such that $s \in \mu_1^{-1}(\bar{c})$.

(*i*) Suppose $\bar{c}$ does not release s in level 2. Then, since c offered admission to s in level 1 with stipend $\min\{m^c, B^c - \sum_{k \in \mu_1^{-1}(c)(s)} x_k^c\}$ and s rejected this offer and accepted the $(\bar{c}, x_s^{\bar{c}})$. Then, it is not possible that

$$(c,x)\ P_s\ (\bar{c}, x_s^{\bar{c}}),\quad \text{where}$$

$$x \leq \min\{m^c, \sum_{s' \in \bar{S}} x_{s'}^c + B^c - \sum_{s'' \in \mu^{-1}(c)} x_{s''}^c\} \leq \min\{m^c, B^c - \sum_{s' \in \mu_1^{-1}(c)(s)} x_{s'}^c\}.$$

This contradicts our assumption that the s prefers the c with the stipend x to his match at μ.

(*ii*) Suppose $\bar{c}$ releases s at some step in level 2. But by the same reasoning as in part (2) in Sub-subcase 1-1-1, we obtain a contradiction.

(2) There is no $\bar{c} \in \mathcal{C} \backslash c$ such that $s \in \mu_1^{-1}(\bar{c})$, that is, $s \in U_1$. By the same reasoning as in Sub-subcase 1-1-2, we obtain a contradiction.

Sub-subcase 1-2-2: There is no $s' \in \mu_1^{-1}(c)$ such that $v_{s'}^c > v_s^c$ and there is $s'' \in \mu^{-1}(c)$ such that $v_{s''}^c > v_s^c$.

Since $v_{s''}^c > v_s^c$ and there is no $s' \in \mu_1^{-1}(c)$ such that $v_{s'}^c > v_s^c$, we have s'' admitted in level 2. Since $v_{s''}^c > v_s^c$, student s receives offer before s''. Then, one of two cases may happen

(1) In level 2, c couldn't win the right to offer admission to s. Then by similar reasoning as in part (2) in Sub-subcase 1-1-1, we obtain a contradiction.

(2) College c wins the right to offer admission to s and admits him, but releases him at some later step. Then s finally wasn't admitted to college c either because

(i) College c couldn't win the right to offer admission to s. Then, by similar reasoning as in part (2) in Sub-subcase 1-1-1 we obtain a contradiction. Or

(ii) College c can not offer admission to s because there is no $\bar{S} \subseteq \mu^{-1}(c)$ such that $\sum_{s' \in \bar{S}} v_{s'}^c < v_s^c$. But this contradicts our claim in beginning of the proof that there is $\bar{S} \subseteq \mu^{-1}(c)$ such that $\sum_{s' \in \bar{S}} v_{s'}^c < v_s^c$.

Case 2: $s \in \mu_1^{-1}(c)$, but $s \notin \mu^{-1}(c)$.

Since $s \notin \mu^{-1}(c)$, then it means c releases s at some step in level 2. Then s finally wasn't admitted to c either because

(i) College c couldn't get the right to offer him. Then, by similar reasoning as in part (2) in Sub-subcase 1-1-1 we obtain a contradiction. Or

(ii) College c can not offer admission to s because there is no $\bar{S} \subseteq \mu^{-1}(c)$ such that $\sum_{s' \in \bar{S}} v_{s'}^c < v_s^c$. But this contradicts our claim in beginning of the proof that there is $\bar{S} \subseteq \mu^{-1}(c)$ such that $\sum_{s' \in \bar{S}} v_{s'}^c < v_s^c$.

This completes the proof.

During the proof we do not consider the stipend adjustments made at the end of the algorithm. This is for the following reasons:

(1) For each $c \in \mathcal{C}$, if we have $\{B^c - \sum_{k \in \mu^{-1}(c)} x_k^c\} > 0$, then we should have $\{B^c - \sum_{k \in \mu_1^{-1}(c)} x_k^c\} > 0$. This is because we are not creating extra available money in the second level. Since $\{B^c - \sum_{k \in \mu_1^{-1}(c)} x_k^c\} > 0$, all students in $\mu_1^{-1}(c)$ are receiving the maximum stipend they can. Therefore, by adjusting stipends for the students in $\mu^2(c)$, we do not affect anyone in $\mu_1^{-1}(c)$.

(2) For each $c \in \mathcal{C}$, if we have $\{B^c - \sum_{k \in \mu^{-1}(c)} x_k^c\} = 0$, then there are no adjustments.

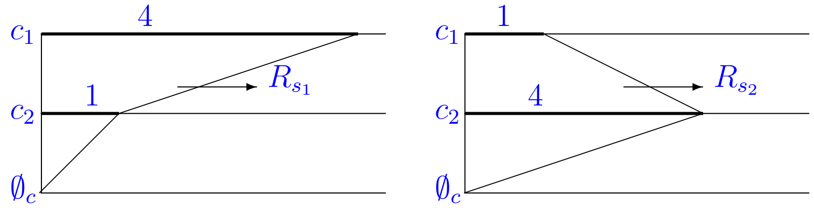
(3) For each $c \in \mathcal{C}$, if we have $\{B^c - \sum_{k \in \mu_1^{-1}(c)} x_k^c\} = 0$, then we cannot have $\{B^c - \sum_{s \in \mu^{-1}(c)} x_s^c\} > 0$, because we are not creating extra available money in the second level. ■

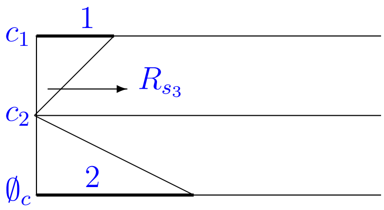
Remark: As you can note, the way we distribute the money left at the end of level 2 of the algorithm is arbitrary. There are many different ways of doing so. Our main result holds for all the possible distribution methods. Result of theorem is valid even if colleges decide to keep the leftover money and not distribute it.

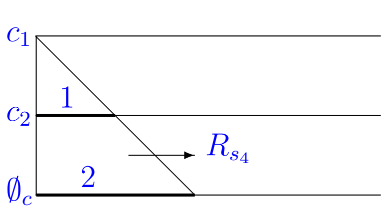
The rule we define in the paper is used to show non-emptiness of the pairwise stable set. But another question that comes to mind is what other properties does this rule satisfy? Efficiency is the one of the properties that immediately comes to mind. Unfortunately we find the negative result even for the weaker notion of efficiency.

Example 6 (The BCF rule is not *weakly Pareto efficient*): Let $\pi \in \Pi$. Let $\mathcal{C} = \{c_1, c_2\}$ and $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$. Let $c_2 \succ c_1$. Let $v_{c_1} = (7, 6, 5, 3)$, $v_{c_2} = (6, 7, 3, 5)$, $B = (7, 7)$, $m = (7, 7)$ and $q = (3, 3)$.

Preferences of students are as follows







Therefore, $\ell^{s_1}(R_{s_1}) = (4,1)$, $\ell^{s_2}(R_{s_2}) = (1,4)$, $\ell^{s_3}(R_{s_3}) = (0,0)$, and $\ell^{s_4}(R_{s_4}) = (0,0)$.

Let's apply Best Comes First algorithm.

Step 1:

We have $O_1^{c_1} = \{\boldsymbol{s_1}, s_2, s_3, s_4\}$ and $O_1^{c_2} = \{s_1, \boldsymbol{s_2}, s_3, s_4\}$. College c_1 offers admission to s_1 with stipend $x_{s_1}^{c_1} = \min\{7,7\} = 7$, and c_2 offers admission to s_2 with stipend $x_{s_2}^{c_2} = \min\{7,7\} = 7$. Both s_1 and s_2 tentatively accept the offers. At the end of step 1 we have $M_1^{c_1} = \{s_1\}$, $R_1^{c_1} = \emptyset$, $M_1^{c_2} = \{s_2\}$, and $R_1^{c_2} = \emptyset$.

Step 2:

We have $O_2^{c_1} = \{\boldsymbol{s_2}, s_3, s_4\}$ and $O_2^{c_2} = \{\boldsymbol{s_1}, s_3, s_4\}$. College c_1 offers admission to s_2 with stipend $x_{s_2}^{c_1} = \min\{7,0\} = 0$, and c_2 offers admission to s_1 with stipend $x_{s_1}^{c_2} = \min\{7,0\} = 0$. Student s_1 compares the offers and since $(c_1, 7)\ P_{s_1}\ (c_2, 0)$, he tentatively accepts the offer of c_1. Similarly, s_2 compares the offers and since $(c_2, 7)\ P_{s_2}\ (c_1, 0)$, he tentatively accepts the offer of c_2. At the end of step 2 we have $M_2^{c_1} = \{s_1\}$, $R_2^{c_1} = \{s_2\}$, $M_2^{c_2} = \{s_2\}$, $R_2^{c_2} = \{s_1\}$.

Step 3:

We have $O_3^{c_1} = \{\boldsymbol{s_3}, s_4\}$ and $O_3^{c_2} = \{s_3, \boldsymbol{s_4}\}$. College c_1 offers admission to s_3 with stipend $x_{s_3}^{c_1} = \min\{7,0\} = 0$, and c_2 offers admission to s_4 with stipend $x_{s_4}^{c_2} = \min\{7,0\} = 0$. Both s_3 and s_4 tentatively accept the offers. At the end of step 3 we have $M_3^{c_1} = \{s_1, s_3\}$, $R_3^{c_1} = \{s_2\}$, $M_3^{c_2} = \{s_2, s_4\}$, $R_3^{c_2} = \{s_1\}$.

Step 4:

We have $O_4^{c_1} = \{\boldsymbol{s_4}\}$ and $O_4^{c_2} = \{\boldsymbol{s_3}\}$. College c_1 offers admission to s_4 with stipend

$x_{s_4}^{c_1} = \min\{7, 0\} = 0$, and c_2 offers admission to s_3 with stipend $x_{s_3}^{c_2} = \min\{7, 0\} = 0$. Student s_3 compares the offers and since $(c_2, 0)\ P_{s_3}\ (c_1, 0)$, he tentatively accepts the offer of c_2. Similarly, s_4 compares the offers and since because $(c_1, 0)\ P_{s_4}\ (c_2, 0)$, he tentatively accepts the offer of c_1. At the end of step 4 we have $M_4^{c_1} = \{s_1\}$, $R_4^{c_1} = \{s_2, s_3\}$, $M_4^{c_2} = \{s_2\}$, $R_4^{c_2} = \{s_1, s_4\}$.

Step 5:

We have $O_5^{c_1} = \{\boldsymbol{s_4}\}$ and $O_5^{c_2} = \{\boldsymbol{s_3}\}$. College c_1 offers admission to s_4 with stipend $x_{s_4}^{c_1} = \min\{7, 0\} = 0$, and c_2 offers admission to s_3 with stipend $x_{s_3}^{c_2} = \min\{7, 0\} = 0$. Both s_3 and s_4 tentatively accept the offers. At the end of step 5 we have $M_5^{c_1} = \{s_1, s_4\}$, $R_5^{c_1} = \{s_2, s_3\}$, $M_5^{c_2} = \{s_2, s_3\}$, $R_5^{c_2} = \{s_1, s_4\}$.

Since both colleges have no student to offer admission to, algorithm stops. The final allocation is $BCF(\pi) = \{(s_1, c_1, 7), (s_2, c_2, 7), (s_3, c_2, 0), (s_4, c_1, 0)\}$.

Now consider allocation $(\mu', x') \equiv \{(s_1, c_2, 5), (s_2, c_1, 5), (s_3, c_1, 2), (s_4, c_2, 2)\}\}$

One can easily check that for each $s \in \mathcal{S}$, we have $(\mu(s), x_s^{\mu(s)})\ P_s\ (\mu'(s), x_s^{\mu'(s)})$. Also for each $c \in \mathcal{C}$, we have $\sum_{s \in \mu^{-1}(c)} v_s^c < \sum_{s \in \mu'^{-1}(c)} v_s^c$. Therefore, for each $s \in \mathcal{S}$ and each $c \in \mathcal{C}$ allocation $(\mu', x') = \{(c_1, s_2, 5), (c_1, s_3, 2), (c_2, s_1, 5), (c_2, s_4, 2)\}$ Pareto dominates the $BCF(\pi) = \{(c_1, s_1, 7), (c_1, s_4, 0), (c_2, s_2, 7), (c_2, s_3, 0)\}$. This shows that the rule associated with Best Comes First Algorithm is not weakly Pareto-efficient.

Although our rule is immune to deviations by a pair of a college and a student, the group of more than one college and one student can benefit from joint deviations. This is the main reason why we get inefficiencies. Therefore, the question that whether we can always find an allocation that is immune to deviations by any group of students and colleges remains still open for this model.

Comments on relation with "matching with contracts" model.

Recently, some papers study matching with general contracts. Necessary and sufficient condition for existence of stable allocation, called substitutability, was introduced (Hatfield and Milgrom (2005)). But later it was shown that this condition is only necessary condition. A weaker condition, called bilateral substitutability, was introduced and it was shown that it is necessary and sufficient condition for existence of stable allocation (Hatfield and Kojima (2008, 2010)).

In words, contracts are substitutes for a college if addition of a contract to the choice set never induces a college to take a contract it previously rejected. In bilateral substitutes the condition is subjected to the case where the student in previously rejected and newly added contracts should not be in any contract in the choice set. We will give an example which shows that colleges' preferences in our model does not satisfy any of these conditions.

Formally, let $a \equiv (s, x_s) \in \mathcal{S} \times \mathbb{R}_+$ be a contract. Let $\mathcal{A}$ be the set of all contracts and let $A \in 2^{|\mathcal{A}|}$. For each contract $a \in \mathcal{A}$, let $s(a)$ be the student in contract a. For a set of contracts A, let $S(A)$ be the set of students in contracts in A. Let $Ch_c : \mathcal{A} \to \mathcal{A}$ be a choice function of college c. Let $Ch_c(A)$ be the choice of college c from the set of contracts A such that $Ch_c(A) \in \{A' \subseteq A \text{ such that } A' \equiv \arg\max_{A'' \subseteq A} \sum_{s \in S(A')} v_s^c\}$. Contracts are bilateral substitutes for a college c if there do not exist contracts a, a' and a set of contracts A such that $s(a), s(a') \notin S(A)$, $a \notin Ch_c(A \cup a)$ and $a \in Ch(A \cup a, a')$.

Next, we provide an example in which we show that budget constraint is not encompassed in feasibility constraints of contracts.

Example 7: Let $S = \{s, s', s''\}$ and $C = \{c, c'\}$. Values of college c are $v^c = \{v_s^c = 7, v_{s'}^c = 5, v_{s''}^c = 4\}$ and its budget is $B^c = 8$. Consider set of contracts $\{(s, 6), (s', 4)\}$. College c cannot choose both contracts due to its budget constraint. Since it values s more than s', then it chooses contract $(s, 6)$ and rejects $(s', 4)$. Now let us add another contract $(s'', 4)$ to the set of contracts. Again, due to its budget constraint, c cannot choose all three contracts. College c also cannot choose any pair of contracts that includes contract $(s, 6)$. Since the values of the students s' and s'' are greater than the value of student s, then it chooses contracts $(s', 4)$ and $(s'', 4)$, and reject $(s, 6)$. But then we get violation of bilateral substitutability condition. Addition of a contract with a student who is not in any previously available contracts, results in college choosing the contract that it was previously rejecting (the student in this contract is not in any other contract as well). And by our result we know that pairwise stable allocation exists for such a problem.

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