Probabilistic and Spatial Models of Voting*

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(abstract)

This paper surveys some models of committees and elections. It begins with a discussion of the history of the models. This part covers influential work by Duncan Black and Harold Hotelling. This discussion also has material about important work by Anthony Downs which built on Hotelling's work. The models developed by Black, Hotelling and Downs were the "pioneering work" for the research being surveyed -- in the sense that the authors who have developed the other models discussed in this survey are authors who have used similar assumptions. The rest of the paper considers assumptions that are similar to the ones used in the "pioneering work". As in some of the subsequent work on this topic, the assumptions are stated in the language of game theory. As these assumptions are set out, the paper discusses work that various other authors have done on models of committees and elections. The paper includes applications of both (1) solution concepts based on the concept of the core from cooperative game theory and (2) solution concepts from non-cooperative game theory.

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This chapter discusses results about committees which use majority rule, some related results about majority rule as a social choice rule (that is, without referring to a specific context where majority rule is used) and results about electoral competitions where the winner is selected by majority rule.

1. Black's analysis of committees & related results about majority rule

An appealing approach to social choice is to select an alternative only if it can't be beaten in pairwise comparisons made with simple majority rule. However (as Condorcet (1785) first discovered) there is a serious problem with this approach: When there are three or more alternatives, there exist preference orderings for voters where each alternative can be beaten.

Black (1948a, 1948b, 1958) developed a model of a committee which uses simple majority rule, and identified assumptions (about the committee members' preferences) which imply that there is an unbeaten alternative in that context. He also characterized the unbeaten alternatives. One of the things this Section does is: Summarize the key elements of Black's model of committees and describe important results he derived in that context.

A number of subsequent references (see, for instance, Mas-Colell, Whinston and Green (1995) and the references they cite) have used the more general approach of 1) considering a model that doesn't specify the context where simple majority rule will be used and 2) identifying results for the model which follow from Black's reasoning. A second thing this Section does is: Illustrate this approach with one such model. Results obtained this way are ones which can be applied to committees *and* other settings where simple majority rule is used.

Black (1948a, 1949, 1958) and Bowen (1943) are early sources that identified other settings which are relevant. Black pointed out that results similar to the ones he obtained for committees hold for an analogous model of multi-candidate elections where 1) each pair of candidates is compared using simple majority rule and 2) the characteristics of the candidates which matter to the voters (e.g., policies embodied by a candidate) are exogenous to the model. In independent work, Bowen developed a model of a community that uses referenda to determine the value of one variable: How much public education will be provided (with tax-sharing rules being given). In his model, he used standard assumptions about an individual's trade-off between public education and private consumption. Bowen assumed that the outcome of the referendum is determined by simple majority rule -- and (within the specific context of his model) obtained results which are similar to Black's results for committees.

1.a) Voters, alternatives and preferences

Black defined a "committee" to be a group of people who arrive at a decision by means of voting. In addition, he assumed that the number of committee members is finite. Black defined a "motion" to be a proposal before a committee, which it may adopt or reject by a method of voting. Black assumed there is a set of motions which have been put forward. Black also assumed each committee member has a weak preference ordering on the set of motions.

For the general model: Assume there is a finite set of voters. Ω will be an index set for these voters. The elements of Ω will be the successive integers $1,...,\#\Omega$. Assume there is a set of alternatives. This set will be denoted by X. Assume that each voter has a preference ordering on X. More specifically, for each $\omega \in \Omega$, there is a preference relation, $\succeq_\omega$, on X which is (i) complete (for all distinct $x, y \in X$: $x \succeq_\omega y$ or $y \succeq_\omega x$), (ii) reflexive (for all $x \in X$: $x \succeq_\omega x$) and (iii) transitive (for all $x, y, z \in X$: $[x \succeq_\omega y \;\&\; y \succeq_\omega z] \Rightarrow [x \succeq_\omega z]$). The asymmetric part of $\succeq_\omega$ will be denoted by $\succ_\omega$. The symmetric part of $\succeq_\omega$ will be denoted by $\sim_\omega$. $x \succeq_\omega y$ will be interpreted as "For voter ω , x is at least as good as y"; $x \succ_\omega y$ will be interpreted as "voter ω prefers x to y"; $x \sim_\omega y$ will be interpreted as "voter ω is indifferent between x and y". A set of alternatives and the preferences for a set of voters will be summarized by $(X, (\succeq_1,...,\succeq_{\#\Omega}))$

1.b) Simple majority rule

Black considered a committee which places each of the motions against every other motion in a vote -- comparing each pair with simple majority rule.

For any given $(X, (\succeq_1,...,\succeq_{\#\Omega}))$, the relation "beats or ties" for simple majority rule will be denoted by $\succeq^S$. More specifically: For each $x, y \in X$, $x \succeq^S y$ means $\#\{\omega \in \Omega\text{: } x \succeq_\omega y\} \geq \#\{\omega \in \Omega\text{: } y \succeq_\omega x\}$. The asymmetric part of $\succeq^S$ will be denoted by $\succ^S$. The symmetric part will be denoted by $\sim^S$. $x \succ^S y$ means x beats y (when simple majority rule is used); $x \sim^S y$ means x ties y (when simple majority rule is used). The definition of $\succeq^S$ clearly implies: (i) For each $x, y \in X$, $x \succ^S y$ if and only if $\#\{\omega \in \Omega\text{: } x \succ_\omega y\} > \#\{\omega \in \Omega\text{: } y \succ_\omega x\}$; (ii) For each $x, y \in X$, $x \sim^S y$ if and only if $\#\{\omega \in \Omega\text{: } x \succ_\omega y\} = \#\{\omega \in \Omega\text{: } y \succ_\omega x\}$.

For any given $(X, (\succeq_1,...,\succeq_{\#\Omega}))$, saying that $x \in X$ is a "(weak) simple majority Condorcet winner" means x is a greatest element for $\succeq^S$ on X (that is, $x \succeq^S y,\ \forall\ y \in X$). [Note: This is equivalent to saying x a maximal element for $\succ^S$ on X (that is, $\nexists\, y \in X$ such that $y \succ^S x$)]. Saying that $x \in X$ is a "(strong) simple majority Condorcet winner" means $x \succ^S y,\ \forall\, y \in X - \{x\}$. [Note: This is equivalent to saying $\nexists\, y \in X - \{x\}$ such that $y \succeq^S x$)]. Using this terminology, the observation at the end of the first paragraph in Section 1 can be re-stated as: If $\#X \geq 3$, then $\exists$ $(\succeq_1,...,\succeq_{\#\Omega})$ where there is no (weak) simple majority Condorcet winner. Of course (a fortiori): If $\#X \geq 3$, then $\exists$ $(\succeq_1,...,\succeq_{\#\Omega})$ where there is no (strong) simple majority Condorcet winner.

1.c) Single-peakedness and median alternatives

Black assumed each committee member's preference ordering is represented by an ordinal utility function. In addition, he considered a setting where there is a one-to-one function from the motions to points on a horizontal axis and there is a vertical axis for ordinal utility.

Black considered committees where either (i) the set of motions is finite or (ii) the set of points assigned to motions is a segment of the horizontal axis. For (i): A piecewise-linear curve was obtained for a utility function by putting a dot at every point in a plane where the horizontal coordinate is the number for a point on the horizontal axis assigned to a motion and the vertical coordinate is the utility number for that motion -- and joining the dots for consecutive motions with segments. For (ii): He assumed the graph for a utility function is a smooth curve.

Black defined a "single-peaked curve" to be a curve which is always upward-sloping, always downward-sloping or upward-sloping to a particular point and downward-sloping beyond it. He assumed the one-to-one function from the motions to a horizontal axis is a function where the resulting curves for the committee member's' utility functions are single-peaked curves.

For the general model: The alternatives will be "lined up" by using a type of ordering relation on X that Rubin (1967) and Denzau and Parks (1975) have called a "linear order" -- more specifically: a relation, $\leq_o$, on X which is (i) complete (for all distinct x, y $\in$ X: $x \leq_o y$ or $y \leq_o x$), (ii) transitive (for all x, y, z $\in$ X: $[x \leq_o y \;\&\; y \leq_o z] \Rightarrow [x \leq_o z]$), and (iii) antisymmetric (for all x, y $\in$ X, $[x \leq_o y \;\&\; y \leq_o x] \Rightarrow$ [x and y are not distinct]). The asymmetric part of $\leq_o$ will be denoted by $<_o$. Black's function (from the motions to points on a horizontal axis) which has the utility functions be single-peaked curves provides a "natural linear order" for the motions -- where, for each pair of motions, $[x \leq_o y] \Leftrightarrow$ [the point on the horizontal axis assigned to x *is the same as or to the left of* the point on the horizontal axis assigned to motion y].

In Black's model, each committee member has a motion which he prefers to every other motion. When each voter in the general model has a unique most-preferred alternative, it will be called a "regular model". More precisely: Any given $(X, (\succeq_1,\ldots,\succeq_{\#\Omega}))$ is a "regular model" if and only if there exists a function

$$m: \Omega \to X$$

where, for each $\omega \in \Omega$, we have $[m(\omega) \succ_\omega y, \;\forall\, y \in X - \{m(\omega)\}]$.

The term "single-peaked" will be used for a property that a voter's preferences can have in a regular model. In particular: For any given $\omega \in \Omega$, saying "$\succeq_\omega$ is single-peaked (with respect to $\leq_o$)" will mean that, for each pair y, z $\in$ X,

$$[[z <_o y \;\&\; y <_o m(\omega)] \;\text{ or }\; [m(\omega) <_o y \;\&\; y <_o z]] \Rightarrow [y \succ_\omega z].$$

In some contexts, X is taken to be a segment of the real line (e.g, when the alternative to be selected is the level of expenditure for a publicly-provided good and that level is modelled as a continuous variable) and a regular model is used. In these contexts, it is natural to consider the ordering relation "is less than or equal to" for real numbers. Under these assumptions, saying that $\succeq_\omega$ is single-peaked (with respect to $\leq$) means that, for each y, z $\in$ X, we have $y \succ_\omega z$ whenever either $[z < y < m(\omega)]$ or $[z > y > m(\omega)]$. In these contexts, it is also common to assume that $\succeq_\omega$ is continuous. When these assumptions hold: [$\succeq_\omega$ is single-peaked with respect to $\leq$]$\Leftrightarrow$ [$\succeq_\omega$ is strictly convex] $\Leftrightarrow$ [$\succeq_\omega$ can be represented by a strictly quasi-concave utility function] (see, for instance, Mas-Colell, Whinston and Green (1995: p. 801)).

When a given (X, ($\succeq_1$,...,$\succeq_{\#\Omega}$)) is a regular model, the term "distribution of most-preferred alternatives" will be used for the probability distribution on X where

$$\Pr(y) = \#\{\omega \in \Omega\text{: } m(\omega) = y\}/\#\Omega, \;\; \forall\, y \in X.$$

Saying that $x \in X$ is a median for this distribution (*with respect to* a particular ordering relation, $\leq_o$) means

$$\Pr(\{y \in X\text{: } y \leq_o x\}) \geq ½ \;\text{ and }\; \Pr(\{y \in X\text{: } x \leq_o y\}) \geq ½.$$

Straightforward applications of these concepts will also be used in stating results that Black proved for his committee model.

1.d) Results for committees and for the general model

Black (1948a: pp. 27-28; 1948b: pp. 249-251; 1958: pp. 16-18) proved the following results (under the assumptions for his committee model which are stated above). First: A particular motion can't be beaten (that is, no motion gets a simple majority over that particular motion) if and only if that particular motion is a median for the distribution of most-preferred motions (*with respect to* the natural linear order). Second: A particular motion gets a simple majority over every other motion if and only if that particular motion is the unique median for the distribution of most-preferred motions (*with respect to* the natural linear order).

The following (analogous) theorem holds under the assumptions for the general model [see, for instance, Denzau and Parks (1975): Theorem 1].

Theorem: Suppose (X, ($\succeq_1$,...,$\succeq_{\#\Omega}$)) is a regular model and there is an ordering relation $\leq_o$ which is such that, for each $\omega \in \Omega$, the preference ordering $\succeq_\omega$ is single-peaked (*with respect to* $\leq_o$). Then
1. $x \in X$ is a (weak) simple majority Condorcet winner if and only if x is a median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$);
2. $x \in X$ is a (strong) simple majority Condorcet winner if and only if x is the unique median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$).

For some committees, an absolute majority (that is, more than half of the entire set of voters) is required for social preference – instead of a simple majority being required. More specifically, for some committees: (i) one alternative beats another one if and only if an absolute majority prefer it and (ii) otherwise, the two alternatives tie one another.

For any given (X, ($\succeq_1$,...,$\succeq_{\#\Omega}$), the relation "beats or ties" for absolute majority rule will be denoted by $\succeq^A$. More specifically: For each x, $y \in X$, $x \succeq^A y$ means $\#\{\omega \in \Omega\text{: } x \succeq_\omega y\} \geq \Omega/2$.

The asymmetric part of $\succeq^A$ will be denoted by $\succ^A$. The symmetric part will be denoted by $\sim^A$. The definition of $\succeq^A$ implies: (i) For each $x, y \in X$, $x \succ^A y$ if and only if $\#\{\omega \in \Omega\text{: } x \succ_\omega y\} > \Omega/2$; (ii) For each $x, y \in X$, $x \sim^A y$ if and only if $\#\{\omega \in \Omega\text{: } x \succ_\omega y\} \leq \Omega/2$ & $\#\{\omega \in \Omega\text{: } y \succ_\omega x\} \leq \Omega/2$. For any given $(X, (\succeq_1,...,\succeq_{\#\Omega}))$, saying that $x \in X$ is a "(weak) absolute majority Condorcet winner" means $x \succeq^A y, \forall\, y \in X$. Saying that $x \in X$ is a "(strong) absolute majority Condorcet winner" means $x \succ^A y, \forall\, y \in X$.

The definitions clearly imply: (1) If an alternative is a (strong) absolute majority Condorcet winner, then it is a (strong) simple majority Condorcet winner; (2) If an alternative is a (weak) simple majority Condorcet winner, then it is a (weak) absolute majority Condorcet winner. The definitions also clearly imply that the converses of those statements are not true *in general*. So, *in general*, results for simple majority Condorcet winners can't be expected to hold for absolute majority Condorcet winners. Nonetheless, the following variation on the previous theorem does hold (see, for instance, Denzau and Parks (1975: Theorem 2)).

Theorem: Suppose $(X, (\succeq_1,...,\succeq_{\#\Omega}))$ is a regular model and there is an ordering relation $\leq_o$ which is such that, for each $\omega \in \Omega$, the preference ordering $\succeq_\omega$ is single-peaked (*with respect to* $\leq_o$). Then
1. $x \in X$ is a (strong) absolute majority Condorcet winner if and only if x is the unique median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$);
2. $x \in X$ is a (weak) absolute majority Condorcet winner if and only if x is a median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$).

It should be noted that there is an important connection between (a) the set of (weak) absolute majority Condorcet winners and (b) a solution concept that has been applied when committee decisions have been modeled as cooperative games (as in the models developed by Kramer and Klevorick (1974) and Nakamura (1975)). If a committee uses absolute majority rule, then a "winning coalition" is a set of voters that contains more than half of the committee members. Saying that alternative x "dominates" alternative y means there exists a winning coalition, W, where $x \succ_\omega y, \forall\, \omega \in W$. Using this dominance relation, we can now easily state the definition of the relevant solution concept: The "core" is the set $\{y \in X : \nexists\, x \in X$ such that x dominates y$\}$. Using this definition, it follows that (when a committee uses absolute majority rule) an alternative is in the core if and only if it is a (weak) absolute majority Condorcet winner. Discussions of part 2 of the above theorem in terms of the core are in McKelvey (1990: Sections 2 & 3.1), Straffin (1994: Section 6), Saari (2004: Sections 3.1 and 4.1) and other references.

The theorems stated above are very significant for settings where either simple majority rule or absolute majority rule is used *and* the supposition for the theorems is satisfied .

To begin with, the theorems are *positive* results for social choice. More specifically, there is at least one median -- so there is at least one (weak) Condorcet winner. So, it is *possible* to select an alternative which can't be beaten.

If $\#\Omega$ is odd, there is a unique median. Hence, if $\#\Omega$ is odd, there is a (strong) Condorcet winner -- which, by definition, is also a unique (weak) Condorcet winner. For the cases where $\#\Omega$ is even: 1) If there is a unique median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$), then there is a unique (weak) Condorcet winner *and* it is also a (strong) Condorcet winner; 2) If there isn't a unique median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$), then there is more than one (weak) Condorcet winner *and* there is no (strong) Condorcet winner.

The ordering relation referred to in the theorems reflects "a similar attitude toward the alternatives" among the voters (since, when this ordering relation is used, every voter has the view that moving away from his most-preferred alternative in either "direction" leads to worse and worse alternatives). Therefore, since the Condorcet winners are medians with respect to that ordering relation, they match with a reasonable measure for the center of the distribution of most-preferred alternatives. Hence they are (in this sense) "centrist" social choices.

The theorems also tell us that one wouldn't have to actually make all the majority rule comparisons to find the appropriate choice(s). Instead, one could find the appropriate choice(s) by finding the median(s) of the distribution of most-preferred alternatives.

An individual whose most-preferred alternative is a median for the distribution of most-preferred alternatives (with respect to a given ordering relation $\leq_o$) is called a "median voter" (with respect to $\leq_o$). When there is a unique median for the distribution of most-preferred alternatives (with respect to $\leq_o$), there will be at least one median voter – and there will be more than one if and only if the median is the most-preferred alternative for more than one individual. In the cases where there is a unique median) the theorems imply that an alternative is a Condorcet winner *if and only if* it is a median voter's most-preferred alternative.

When there is more than one median for the distribution of most-preferred alternatives (with respect to $\leq_o$), there will be at least two median voters – and there will be more than two if and only if there is a median which is the most-preferred alternative for more than one individual. In the cases where there is more than one median, the theorems imply that an alternative is a (weak) Condorcet winner *if* it is a median voter's most-preferred alternative. In these cases, there will always be two distinct (weak) Condorcet winners which are most-preferred alternatives for median voters. In addition, in some instances, there will also be at least one (weak) Condorcet winner which is not anyone's most-preferred alternative. More specifically: Let α be the index number for an individual who has one of the two distinct most-preferred alternatives which are (weak) Condorcet winners as his most-preferred alternative, let β be the index number for an individual who has the other of the two alternatives as his most-preferred alternative and have the index numbers be such that $m(\alpha) <_o m(\beta)$; The theorems imply that any alternative, y, which satisfies $m(\alpha) <_o y <_o m(\beta)$ will be a (weak) Condorcet winner.

Related discussions of single-peakedness and majority rule are in Arrow (1963: Section 2 in Chapter VII), Fishburn (1973: Sections 9.1-9.3), Denzau and Parks (1975: Sections 1 & 2), Enelow and Hinich (1984: Sections 2.1 & 2.2), Mueller (2003: The last paragraph in Section 5.2 & all of Section 5.3), MasColell, Whinston and Green (1995: Section 21.D), Shepsle and Bonchek (1997: pp. 83-90 in the section "The Simple Geometry of Voting") and other references.

2. Hotelling, Downs & electoral competition

An electoral competition has voters with preferences on alternatives, like the models discussed in Section 1. In addition, it has political candidates (or parties) who *compete* with each other by choosing alternatives. After each candidate chooses an alternative (which will become the social choice if he is elected), the voters vote on the candidates and those votes determine the outcome of the election. One important feature of an electoral competition is: Unlike in the treatment of voting in Section 1, the voters do not vote directly on the elements in the set of alternatives -- instead, they vote directly on candidates. Another important feature is: The characteristics of a candidate which matter to voters (e.g., policies embodied by a candidate) are endogenous.

The literature on electoral competition began with Hotelling (1929). In that paper, Hotelling introduced an influential model for duopolists (which includes, as a special case, a locational model). Near the end, he briefly described how his locational model could be reinterpreted as a model of competition between two political parties. He then concluded that, in his reinterpreted model, there's a tendency for the competing parties to imitate each other.

Downs (1957) also played a crucial role in the development of the literature on electoral competition -- by taking the step of stating *explicit assumptions* which correspond to Hotelling's brief description of his reinterpreted model. The explicit model he specified will be called the "Hotelling-Downs model".

Subsection 2.a) will describe Hotelling's model for duopolists. Subsection 2.b) will discuss how Hotelling's locational duopoly model has been reinterpreted as a model of electoral competition.

2.a) Hotelling's model for duopolists

Hotelling (1929) considered duopolists who offer an identical commodity (which comes in indivisible units). For each of the two sellers, production is costless. There is also a set of buyers. Each buyer will purchase one unit of the product. A buyer must purchase his unit from one of the duopolists.

Each buyer has a home. The location of a home is modeled as a point in a segment of the real line (which could be, for instance, Main Street in a town or a transcontinental railroad). It is assumed that there is a continuous uniform distribution of buyers' homes along the segment. Each duopolist has one store, and every purchase must be made at one of these stores. The possible locations for a store are modeled as the points in the segment.

Each duopolist's objective is to maximize its profits. Each buyer will pay a price for the unit he purchases, and will transport it home at a constant per-unit-travelled cost. The per-unit-travelled cost is the same for each buyer. Each buyer's objective is to minimize the total of price plus transportation cost.

In his paper, Hotelling devoted a lot of attention to situations where each duopolist can choose a price which it charges its own customers. In addition, he considered situations where the duopolists do not have any control over the price in their industry, but the stores are thought of as movable. The resulting locational model (which has locational competition, but no price competition) is relevant for this chapter, because this is the specific model that Hotelling suggested reinterpreting as a model of electoral competition.

In Hotelling's locational model for duopolists: Both duopolists charge their customers the same price (with the price being exogenous to the model), but each duopolist can choose its own location for its store before the buyers make their purchases. Under the assumptions for the locational model, (1) a duopolist will achieve its objective by choosing a location (for its store) which maximizes its market share and (2) a buyer will achieve his objective by purchasing his unit of the product at a store which minimizes his transportation cost.

In Hotelling's locational model, the firms can "look ahead" and determine how the buyers would respond to their possible choices of store locations. When they do, the duopolists' decisions define a two-person game. The unique pure-strategy equilibrum for this game is: Both duopolists locate at the center of the segment.

Hotelling also identified an important welfare property of the equilibrium -- by comparing the equilibrium locations with the locations that will minimize the total of transportation charges paid by the consumers. He concluded that the social objective of minimizing the total transportation charges is achieved if and only if the sellers occupy symmetrical positions at the quartiles -- instead of locating at the center of the segment. So, in equilibrium, this social objective is *not* achieved by the duopolistic competition.

2.b) Interpretation as a model of electoral competition

This Subsection will discuss how the various parts of Hotelling's locational model for duopolists have been interpreted in a model of electoral competition.

In the Hotelling-Downs model of electoral competition, the duopolists are relabelled as two political candidates. The buyers are interpreted as voters. A duopolist offering a commodity to buyers is reinterpreted as the corresponding candidate giving voters an opportunity to vote for him. A buyer purchasing a unit of the product from a duopolist is interpreted as a voter casting a vote for the corresponding candidate. Accordingly, each voter must cast one (and only one) vote and he must cast it for one of the two candidates.

The locations in the line segment become the alternatives. As in Section 1, the set of alternatives will be denoted by X. A buyer's home is relabelled as corresponding voter's most-preferred alternative. It is assumed that there is a continuous uniform distribution of voters' most-preferred alternatives along the segment.

Downs suggested that, under certain circumstances, one could think about the possible "locations" for the political candidates as positions on a left-right political spectrum (where, starting at any point in the interior 1) the more you move to the left the more liberal the position and 2) the more you move to the right the more conservative the position.) .

Downs also suggested that, in some of these cases, the set of possible locations (or candidate positions) could be taken to be the segment on the real number line from zero to 100 -- with a candidate's position indicating the percentage of the economy it wants left in private hands. He argued that this particular interpretation would be plausible if, for instance, voters are solely concerned about the amount of government intervention in the economy.

A firm's store location is relabelled as the alternative that the corresponding candidate chooses. A candidate can choose any point in the line segment. In Hotelling's locational model for duopolists, a firm maximizes profits if and only if it maximizes the percentage of the buyers who purchase its product (that is, its market share). Each firm's objective is (accordingly) re-interpreted as the corresponding candidate maximizing the percentage of the voters who vote for him (that is, his vote share).

Hotelling (1929) assumed that the transportation cost to a buyer from making his purchase at a firm's store is the following product: (per-unit-travelled cost)·(the distance *from* the firm's store *to* the buyer's location). For the political interpretation suggested by Hotelling, one is substituting costs of distance in the segment of alternatives for transportation costs. So the total cost to a voter from voting for a political candidate is the following product: (per-unit cost)·(the distance *from* the candidate's position *to* the voter's most-preferred alternative). Since a voter's most-preferred alternative is fixed and the per-unit cost is fixed, this defines a "total cost function" (for a voter) on the set of possible candidate positions.

Using notation which is similar to what was used in Section 1: For any given ω, the corresponding most-preferred alternative can be written as $m(\omega)$. *If* we also let κ denote the (positive and constant) per-unit cost and let $TC_\omega(x)$ denote the total cost to ω from voting for a candidate with position x, *then* the total cost function for ω can be written as $TC_\omega(x) = (\kappa)\cdot|x - m(\omega)|$. For the political interpretation (of the locational model for duopolists) suggested by Hotelling: For any given $\omega \in \Omega$, $U_\omega(x) = -TC_\omega(x)$ can be thought of as a utility function on X. References which have translated Hotelling's assumption about buyer preferences into this assumption about voter preferences include Davis and Hinich (1966), Stokes (1966), Buchanan (1968), Myerson (1995) and Osborne (1995).

In the duopoly model that Hotelling suggested reinterpreting, a buyer will minimize his transportation cost. So each firm will expect a particular buyer to select a particular firm's store *if* (for that buyer) the transportation cost from buying his unit quantity at that firm's store is less than the transportation cost from buying his unit quantity at the other firm's store. When the transportation costs are equal, the buyer will be indifferent between the two firms. In this case, the firms could expect the buyer's choice to be equivalent to the toss of a fair coin.

For the political interpretation suggested by Hotelling, these assumptions can be re-stated as each party having the following expectations (for any given ω): *If* $TC_\omega(s_1) < TC_\omega(s_2)$, *then* the probability that ω will vote for party 1 is one; *If* $TC_\omega(s_1) > TC_\omega(s_2)$, *then* the probability that ω will vote for party 2 is one; *If* $TC_\omega(s_1) = TC_\omega(s_2)$, *then* the probability for each of ω 's two possible choices is one-half.

Under these assumptions, the candidates' decisions define a two-person game (which will be analyzed in more detail in Section 5). The unique pure-strategy equilibrum for this game is: Both candidates choose the alternative which is at the center of the segment.

The equilibrium behavior of the candidates is directly analogous to the equilibrium behavior of the duopolists. However, the welfare properties of the equilibria are not the same. The next paragraph discusses the candidates' equilibrium from a welfare perspective.

Each voter ends up with the social alternative of the winning candidate rather than the social alternative of whichever candidate he votes for (in contrast to each buyer making his purchase at the specific store that he chooses, instead of all of the buyers making their purchases at the store that a majority prefers). So Buchanan (1968, p. 329) and Myerson (1995, p. 78) have argued that an appropriate social objective is to minimize the total cost of distance from the voters' most-preferred alternatives to the position of the winning candidate. They have concluded that this social objective is achieved if and only if the candidates locate at the center of the segment. So, in equilibrium, this social objective *is* accomplished by the electoral competition.

3. A framework for models of electoral competition

This Section provides a framework that will (in subsequent sections) be useful for stating results from the literature on electoral competition.

3.a) Basic assumptions and notation

I. Some electorates are very large and (as in the work of Hotelling and Downs) it is useful to model the voters as an infinite set. Other electorates are relatively small and it is appropriate to model the voters as a finite set. The following assumptions allow for both possibilities.

There is a set of voters. Ω will be an index set for the voters. $\mathscr{F}(\Omega)$ will be a σ-field of subsets of Ω. $(\Omega, \mathscr{F}(\Omega), \mu_\Omega)$ will be a finite measure space. As (for instance) in McKelvey, Ordeshook and Ungar (1980: p. 162), for each $B \in \mathscr{F}(\Omega)$, $\mu_\Omega(B)$ "represents the size of the coalition B". The same approach is also used in McKelvey and Ordeshook (1976: p. 1173), where they (similarly) state that the finite measure of a set of voters "represents the 'number' of voters in the set."

π_Ω will denote the probability measure on $(\Omega, \mathscr{F}(\Omega))$ which is obtained when we normalize μ_Ω. As (for instance) in Davis, DeGroot and Hinich (1972: p. 149), Sloss (1973: p. 23) McKelvey (1975: pp. 817-818), Grandmont (1978: p. 324) or Calvert (1986: p. 9), for each $B \in \mathscr{F}(\Omega)$, $\pi_\Omega(B)$ is the proportion of the voters with an index in B. π_Ω will be called the "distribution of voter indices".

These assumptions can be illustrated with the Hotelling-Downs model. For that model, Ω can be a closed interval $[\alpha, \beta]$ on the real line. With $\Omega = [\alpha, \beta]$, we can have $\mathscr{F}(\Omega)$ be the collection of Borel sets in $[\alpha, \beta]$. For each $B \in \mathscr{F}(\Omega)$, we can then have $\mu_\Omega(B) = \lambda(B)$ (where λ denotes Lebesgue measure). This gives us $\pi_\Omega(B) = \lambda(B) / (\beta - \alpha)$, for each $B \in \mathscr{F}(\Omega)$. This implies (among other things) that, when we consider a sub-interval of the index set for the voters, the proportion of the voters in that sub-interval *equals* [the length of that sub-interval] / [the length of the index set for the voters]. Since this is a continuous uniform distribution on $[\alpha, \beta]$, we could alternatively use the continuous distribution function $F(\omega) = (\omega - \alpha) / (\beta - \alpha)$ on the set Ω or the continuous density function $f(\omega) = 1 / (\beta - \alpha)$ on the set Ω.

When there is a finite set of voters, the index set will be $\Omega = \{1, \dots, \#\Omega\}$, $\mathscr{F}(\Omega)$ will be the power set of Ω, and μ_Ω will be the counting measure on $(\Omega, \mathscr{F}(\Omega))$. These assumptions imply that, for each $W \subseteq \Omega$, we have $\pi_\Omega(W) = \mu_\Omega(W) / \#\Omega$. Since this is a discrete uniform distribution on Ω, we could alternatively use the discrete distribution function $F(\omega) = \omega / \#\Omega$ on the set Ω or the discrete density function (or probability function) $f(\omega) = 1 / \#\Omega$ on the set Ω.

II. As in the model in Section 1: There is a set of alternatives, X, and each voter has a preference ordering on X. For each $\omega \in \Omega$, the preference ordering is denoted by $\succeq_\omega$. The asymmetric part of $\succeq_\omega$ is denoted by $\succ_\omega$, and the symmetric part of $\succeq_\omega$ is denoted by $\sim_\omega$.

Within this framework, saying we have a regular model will mean there is a function

$$m: \Omega \to X$$

where 1) for each ω, $x = m(\omega)$ is the most-preferred alternative in X for the voter whose index is ω <u>and</u> 2) m is measurable on $(\Omega, \mathscr{F}(\Omega))$. With a regular model, one can go from the measures for the indices to corresponding measures for the most-preferred elements. Let $(X, \mathscr{F}(X))$ be a measurable space where, for each $A \in \mathscr{F}(X)$, the set $\{\omega \in \Omega : m(\omega) \in A\}$ is in $\mathscr{F}(\Omega)$. The finite measure on $(X, \mathscr{F}(X))$ induced by μ_Ω will be denoted by μ_X. The probability measure on (X,

$\mathscr{F}(X)$) induced by π_Ω will be denoted by π_X. The measure π_X will be called the "distribution of most-preferred alternatives".

These features of the framework can also be illustrated with the Hotelling-Downs model. In that model, X is a segment of the real line. The segment will be denoted by [a, b]. We can start with the formulation for the index set described above (where, as we've already seen, π_Ω is a continuous uniform distribution of voter indices). We can let $\mathscr{F}(X)$ be the collection of Borel sets in [a, b]. To go from the distribution of voter indices to the distribution of most-preferred alternatives, we can use

$$m(\omega) = [(a\beta - b\alpha) + (b - a)\,\omega] / (\beta - \alpha)].$$

Then, for each $A \in \mathscr{F}(X)$, we would have $\pi_X(A) = \lambda(A) / (b - a)$. This is a continuous uniform distribution on [a, b]. So the distribution of most-preferred alternatives will have the distribution function $F(x) = x / (b - a)$ on the set X and the density function $f(x) = 1 / (b - a)$ on the set X.

For the Hotelling-Downs model, the uniform distribution of voter indices is transformed into a uniform distribution of most-preferred alternatives. However, the framework also lets us consider other mappings from Ω to X. So the framework clearly allows for many other possible distributions of most-preferred alternatives.

When there is a finite set of voters, going from the measures for the indices to the measures for the most-preferred alternatives will give us the same distribution of most-preferred alternatives as was used in Section 1. That is, (using the notation used in this framework) the distribution of most-preferred alternatives will be the probability distribution on X where

$$\pi_X(\{y\}) = \#\{\omega \in \Omega\colon m(\omega) = y\} / \#\Omega, \ \forall\, y \in X.$$

III. Mueller (2003: p. 180) argues that, in the Hotelling-Downs model, "the words 'candidate' or 'party' can be used interchangeably ... for the implicit assumption when discussing parties is that they take a single position in the voter's eyes." In what follows, I will use these two terms interchangeably.

There will be two political candidates. C = {1, 2} is an index set for them. For each $c \in C$, there is a pure strategy set, S_c. It will be assumed that $S_1 = S_2 = X$. s_c denotes a strategy for a particular $c \in C$.

In this framework, the candidates' "common strategy set" won't have to satisfy the specific assumptions about the possible candidate choices that are in the Hotelling-Downs model. Rather, those assumptions illustrate *one* (influential) way in which candidates' strategy sets have been formulated. This flexibility is being included in the framework because the subsequent literature has considered a variety of assumptions about the candidates' common strategy set.

Downs (1957) interpreted the possible "locations" for political parties as "party ideologies" (see pp. 114-115). Selecting a party ideology is clearly one way in which a party can potentially embody policies -- since a party ideology can indicate (to a voter) the policies that a party plans to implement. Some references have adopted this approach and have explicitly assumed that a strategy for a candidate is something which *indirectly* indicates policies that a party plans to implement (see, for instance, Enelow and Hinich (1984) or Hinich and Munger (1997)).

A second approach has been to assume that a party's strategy *directly* identifies the policies it plans to implement. When this approach has been used, the candidates' "common strategy set" has usually been a set of possible policies (see, for instance, Davis and Hinich (1966, 1968, 1971), Davis, Hinich and Ordeshook (1970), Riker and Ordeshook (1973) or Ordeshook (1986)).

A third approach has been to assume that a candidate might want "to augment his alternative platforms to include obfuscation of the policies that he will adopt if he wins the election" (Ordeshook (1986: p. 180)). When a candidate uses this type of strategy (which Ordeshook (1986) calls a "risky strategy"), he selects a lottery on a set of possible policies and communicates that lottery to the voters -- with the voters then being uncertain about what policies will be implemented. So, when this third approach has been used, the candidates' "common strategy set" has been a set of lotteries (see, for instance, Enelow and Hinich (1984: Section 7.3) or Ordeshook (1986)).

With each of these approaches, the candidates' "common strategy set" has been taken to be a pure strategy set. In this framework, I am (similarly) assuming that the elements of X are pure strategies. [Note: Subsection 11.b), will discuss work that has considered mixed extensions of two-candidate games.]

IV. Arrow (1987: p. 124) has observed that for an "election to an office. ... the candidates ... are evaluated by each voter, and the evaluations lead to messages in the form of votes." Each voter will cast one vote. The voters learn s_1 and s_2 *before* they vote. For each $\omega \in \Omega$, there will be a function

$$Pr^1_\omega: S_1 \times S_2 \to [0,1]$$

where the value assigned to a particular $(s_1, s_2) \in S_1 \times S_2$ by Pr^1_ω is the probability that ω will vote for candidate 1 when candidate 1 chooses s_1 and candidate 2 chooses s_2. The probability that ω will vote for candidate 2 when candidate 1 chooses s_1 and candidate 2 chooses s_2 will be

$$Pr^2_\omega(s_1, s_2) = 1 - Pr^1_\omega(s_1, s_2).$$

These probabilities could be objective probabilities, or they could be subjective probabilities (with each candidate having the same expectations).

For each $c \in C$ and $\omega \in \Omega$, $V^c_\omega(s_1, s_2)$ will be the Bernoulli random variable where 1) a "success" is a vote for c from ω, and 2) the probability of "success" is $Pr^c_\omega(s_1, s_2)$ [that is, $V^c_\omega(s_1, s_2)$ equals 1 with probability $Pr^c_\omega(s_1, s_2)$ and $V^c_\omega(s_1, s_2)$ equals 0 with probability $1 - Pr^c_\omega(s_1, s_2)$].

V. Arrow (1987: p. 124) has pointed out that "for the election to an office ... the social decision, which candidate to elect ..., is made by aggregating the votes according to the particular voting scheme used."

In describing the voting scheme for the Hotelling-Downs model, Downs stated that (as is common in democratic nations, when there are two candidates running for an office): a "single party ... is chosen by popular election to run the government apparatus" (p. 23) and a "party ... receiving the support of a majority of those voting is entitled to take over the powers of government" (p. 24).

For the chapter's framework: I will use B_c to denote the set of voters who vote for c, and I will assume that, for each $c \in C$, we have $B_c \in \mathscr{F}(\Omega)$. The voting scheme will be: (1) If $\mu_\Omega(B_1) > \mu_\Omega(B_2)$, then party 1 wins; (2) If $\mu_\Omega(B_1) < \mu_\Omega(B_2)$, then party 2 wins; (3) If $\mu_\Omega(B_1) = \mu_\Omega(B_2)$, then the two parties tie. When the set of voters is finite, the voting scheme can be stated as: (i) If one of the parties gets more votes than the other party, then the party with more votes wins; (ii) If each party gets the same number of votes, then the two parties tie.

This voting scheme, of course, corresponds to an important collective choice rule – the "method of majority decision" (as defined by Arrow (1963: p. 46)). Theorem 1 (Possibility Theorem for Two Alternatives) in Arrow (1963: p. 48) established that, when there are two alternatives (as in the two-party elections that are under consideration), the method of majority decision satisfies all of his conditions -- and, as he pointed out, "Theorem 1 is, in a sense, the logical foundation of the Anglo-American two-party system." [Arrow (1963: p. 48)]

3.b) Possible objectives

To specify a strategic form game, one has to identify the set of players, the strategy set for each player, and the payoff function for each player. In a model of electoral competition, the players are the political candidates. The framework in Subsection 3.a) provides up with both a set of candidates and a strategy set for each candidate. Hence the framework in Subsection 3.a) together with an objective for each candidate which defines a payoff function on $S_1 \times S_2$ will specify a strategic form game. There are various objectives for the candidates that have been considered in the literature on electoral competition. This Subsection will discuss the most important objectives that have been considered.

A number of references have also considered whether replacing one objective with another would affect the candidates' decisions. So this Subsection will additionally compare some of the objectives (using the game-theoretic notion of strategic equivalence for non-competitive games used in Vorobev (1977)).

The first two objectives will be based on the number of votes that a candidate gets. The nature of a candidate's vote total in an electoral competition model will be illustrated in the next few paragraphs. These paragraphs will focus on models with a finite set of voters. A more general treatment will follow.

By definition, when there is a finite set of voters, the total vote for a candidate will just be the sum of the votes cast for him by the individual voters. One (simple) case occurs when, for a given $c \in C$ and $(s_1, s_2) \in S_1 \times S_2$, $V^c_\omega(s_1, s_2)$ is a constant random variable for each $\omega \in \Omega$. In this case, the total vote for c will simply be the number of voters with $Pr^c_\omega(s_1, s_2) = 1$.

In other cases, we can make use of the fact that (for any given $\omega \in \Omega$, $c \in C$ and $(s_1, s_2) \in S_1 \times S_2$) the random variable $V^c_\omega(s_1, s_2)$ can be thought of as one Bernoulli trial [where a success is a vote for c from ω and the probability of a success is $Pr^c_\omega(s_1, s_2)$] in a sequence of trials — in particular, a sequence where there is one trial in the sequence for each voter.

A relatively simple case occurs when (for a given $c \in C$ and $(s_1, s_2) \in S_1 \times S_2$) we have $Pr^c_\omega(s_1, s_2) \in (0,1)$, $\forall\ \omega \in \Omega$ and $Pr^c_\omega(s_1, s_2)$ has the same value for each $\omega \in \Omega$. In this case, the total vote for c is an ordinary binomial random variable [where the number of trials is equal to the number of voters and the probability of success for each trial is $Pr^c_\omega(s_1, s_2)$]. For instance: Suppose that both candidates choose the same strategy and we have $Pr^c_\omega(s_1, s_2) = 1/2$, $\forall\ \omega \in \Omega$. Then the resulting binomial random variable for c's total vote has $\#\Omega$ trials and the probability of success for each trial is ½.

The assumptions for the framework in Subsection 3.a) also clearly allow the values for the $Pr^c_\omega(s_1, s_2)$ to vary from one voter to another. When $Pr^c_\omega(s_1, s_2) \in (0,1)$, $\forall\ \omega \in \Omega$, but $Pr^c_\omega(s_1, s_2)$ does <u>not</u> have the same value for all voters, the total vote for c is a Poisson binomial random variable (using, for instance, the definition in Section 12.2 of Johnson, Katz and Kemp (1992)).

Since a candidate can be uncertain about which candidate a particular voter will vote for, he can be uncertain about the total number of votes he will get. Because of this potential uncertainty, "theories of election competition frequently use maximizing expected vote" (Aranson, Hinich and Ordeshook (1973: p. 205)). The expected vote for candidate c (when candidate1 chooses s_1 and candidate 2 chooses s_2) will be denoted by $EV^c(s_1, s_2)$.

When the set of voters is finite, the expected vote for c will be the sum of the expected votes from the individual voters. Applying the definition of expected value to the random variable $V^c_\omega(s_1, s_2)$, we get

$$EV^c_\omega(s_1, s_2) = (1)\cdot Pr^c_\omega(s_1, s_2) + (0)\cdot Pr^k_\omega(s_1, s_2). \tag{3.b.1}$$

Therefore, the expected vote for c from ω (when candidate 1 chooses s_1 and candidate 2 chooses s_2) will be $Pr^c_\omega(s_1, s_2)$. So, when the set of voters is finite, at any given $(s_1, s_2) \in S_1 \times S_2$ the expected vote for c can be written as

$$EV^c(s_1, s_2) = \sum_{\omega=1}^{\#\Omega} Pr^c_\omega(s_1, s_2) \tag{3.b.2}$$

It was noted above that, in the case where (for a given $c \in C$ and $(s_1, s_2) \in S_1 \times S_2$) we have $Pr^c_\omega(s_1, s_2) \in (0,1)$, $\forall\, \omega \in \Omega$ and $Pr^c_\omega(s_1, s_2)$ has the same value for each $\omega \in \Omega$, the total vote for c is an ordinary binomial random variable In this case, (3.b.2) reduces to $(Pr^c_\omega(s_1, s_2))\#\Omega$.

When considering objectives for the more general framework in Subsection 3.a), I will assume that, for each $(s_1, s_2) \in S_1 \times S_2$, $Pr^1_\omega(s_1, s_2)$ is measurable with respect to $(\Omega, \mathscr{F}(\Omega))$. Using the fact that $Pr^1_\omega(s_1, s_2)$ is bounded below by 0 and bounded above by 1, it follows that, for each $(s_1, s_2) \in S_1 \times S_2$, $Pr^1_\omega(s_1, s_2)$ is integrable with respect to $(\Omega, \mathscr{F}(\Omega), \pi_\Omega)$. So, for all of the cases covered by the framework in Subsection 3.a), the expected vote for c = 1 is (using the finite measure μ_Ω):

$$EV^1(s_1, s_2) = \int_\Omega Pr^1_\omega(s_1, s_2) d\mu_\Omega(\omega) \tag{3.b.3}$$

(3.b.3) implies

$$EV^2(s_1, s_2) = \mu_\Omega(\Omega) - EV^1(s_1, s_2) = \int_\Omega Pr^2_\omega(s_1, s_2) d\mu_\Omega(\omega) \tag{3.b.4}$$

When each candidate's objective is to maximize his expected vote, we have the strategic form game where: 1) the players are the two candidates; 2) for each $c \in C$, the strategy set is S_c; 3) for each $c \in C$, the payoff function is $EV^c(s_1, s_2)$. This game will be denoted by $(S_1, S_2; EV^1, EV^2)$.

In the Hotelling-Downs model (and in a number of subsequent models), each c's objective is to maximize his vote share. Within the framework in Subsection 3.a), this can be generalized to expected vote share. The expected vote share for a particular c (when candidate 1 chooses s_1 and candidate 2 chooses s_2) will be denoted by $VS^c(s_1, s_2)$. The expected vote share for c = 1 is (using the distribution of voter indices, π_Ω):

$$VS^1(s_1, s_2) = \int_\Omega Pr^1_\omega(s_1, s_2) d\pi_\Omega(\omega) \tag{3.b.5}$$

(3.b.5) implies

$$VS^2(s_1, s_2) = 1 - VS^1(s_1, s_2) = \int_{\Omega} Pr^2_{\omega}(s_1, s_2) d\pi_{\Omega}(\omega) \qquad (3.b.6)$$

When each candidate's objective is to maximize his expected vote share: 1) the players are the two candidates; 2) for each $c \in C$, the strategy set is S_c; 3) for each $c \in C$, the payoff function is $VS^c(s_1, s_2)$. This game will be denoted by $(S_1, S_2; VS^1, VS^2)$.

The game-theoretic notion of strategic equivalence can be used with models of electoral competition as follows. Any pair of relevant games for the candidates can, by definition, differ at most in their payoffs. Consider any such pair of games, $(S_1, S_2; a^1, a^2)$ and $(S_1, S_2; b^1, b^2)$. Applying the standard definition of strategically equivalent games (see, for instance, Vorobev (1977, p. 3)) in this context: Saying that $(S_1, S_2; a^1, a^2)$ and $(S_1, S_2; b^1, b^2)$ are strategically equivalent means $\exists$ a positive real number ρ and real numbers γ_1, γ_2 such that, for each, $(s_1, s_2) \in S_1 \times S_2$,

$$a^1(s_1, s_2) = \gamma_1 + \rho \cdot b^1(s_1, s_2) \quad \& \quad a^2(s_1, s_2) = \gamma_2 + \rho \cdot b^2(s_1, s_2) \qquad (3.b.7)$$

Using the definition of strategically equivalent games along with (3.b.3)-(3.b.6), it follows that $(S_1, S_2; VS^1, VS^2)$ and $(S_1, S_2; EV^1, EV^2)$ are strategically equivalent.

One important implication of the definition of strategically equivalent games is: The games have the same equilibria (see, for instance, Vorobev (1977, p. 3)). So 1) a pair of strategies, $(s_1^*, s_2^*) \in S_1 \times S_2$, is a pure-strategy equilibrium in the game $(S_1, S_2; VS^1, VS^2)$ if and only if (s_1^*, s_2^*) is a pure-strategy equilibrium in the game $(S_1, S_2; EV^1, EV^2)$ and 2) a pair of probability distributions on X is an equilibrium in the mixed extension of $(S_1, S_2; VS^1, VS^2)$ if and only if it is an equilibrium in the mixed extension of $(S_1, S_2; EV^1, EV^2)$.

A second important implication of the definition of strategically equivalent games is: For any constant-sum game, there exists a zero-sum game which is strategically equivalent to it (see, for instance, Vorobev (1977, p. 5)). Since the total vote is always $\mu_{\Omega}(\Omega)$, the expected votes for the two candidates always sum to $\mu_{\Omega}(\Omega)$. Therefore, $(S_1, S_2; EV^1, EV^2)$ is constant-sum. Since the expected vote shares (by definition) always sum to 1, $(S_1, S_2; VS^1, VS^2)$ is also constant-sum. The next game is one which has been studied in the literature and which is strategically equivalent to the games specified above.

In some references, it is assumed that each candidate's objective is to maximize his expected plurality. The expected plurality for candidate c at a particular pair of candidate strategies, $(s_1,s_2) \in S_1 \times S_2$, will be denoted by $P\ell^c(s_1, s_2)$. Within the framework in Subsection 3.a), the expected plurality for c = 1 is:

$$P\ell^1(s_1, s_2) = \int_{\Omega} [Pr^1_{\omega}(s_1, s_2) - Pr^2_{\omega}(s_1, s_2)]\, d\mu_{\Omega}(\omega) \qquad (3.b.8)$$

(3.b.8) implies

$$P\ell^2(s_1, s_2) \;=\; - P\ell^1(s_1, s_2) \qquad (3.b.9)$$

When each candidate's objective is to maximize his expected plurality: 1) the players are the two candidates; 2) for each $c \in C$, the strategy set is S_c; 3) for each $c \in C$, the payoff function is $P\ell^c(s_1, s_2)$. This game will be denoted by $(S_1, S_2; P\ell^1, P\ell^2)$.

The fact that $(S_1, S_2; P\ell^1, P\ell^2)$ is zero-sum follows from (3.b.9). The fact that $(S_1, S_2; P\ell^1, P\ell^2)$ is strategically equivalent to $(S_1, S_2; VS^1, VS^2)$ and $(S_1, S_2; EV^1, EV^2)$ follows from the definition of strategic equivalence and the payoff functions.

For the voting scheme specified at the end of Subsection 3.a), one can also specify the probability of a candidate winning. Some references which have used this voting scheme have considered *both* what happens when each candidate is assumed to maximize his expected plurality *and* what happens under the alternative assumption that each candidate maximizes his probability of winning. There are games which differ by having these two different payoff functions where the candidates will definitely make different decisions (which, of course, implies that they are not strategically equivalent). However, assumptions that imply the same candidate decisions under these two payoff functions have been identified by Aranson, Hinich, and Ordeshook (1973), Hinich (1977), Samuelson (1984), Ordeshook (1986), Lindbeck and Weibull (1987), Patty (2002) and others.

Each objective discussed above is a function of votes. Some other assumptions about the objectives of a political candidate have also been considered. For instance: Wittman (1977), Calvert (1985), Roemer (2001) and others have analyzed models where candidates either are solely interested in an election's policy outcome or are willing to make a tradeoff between an election's policy outcome and the margin of victory. For these models, Calvert (1985) has established that (in most of the cases that have been studied in the literature) "candidate policy motivations don't affect the conclusions of the spatial model" (p. 73). Similar results are in Wittman (1977: Proposition 5) and Roemer (2001: Theorem 2.1). As a consequence, nothing of significance is lost if one assumes that each candidate's objective is to maximize his expected vote (or, equivalently, his expected vote share or expected plurality).

4. Deterministic voting & simple majority rule

4.a) Definitions

The Hotelling-Downs model and some of the other models in the electoral competition literature have assumed that a voter's choices are determined by his preferences on the set of alternatives. More precisely, they have assumed: For each $\omega \in \Omega$,

$$\Pr^1_{\omega}(s_1, s_2) = \begin{cases} 1 \text{ if } s_1 \succ_{\omega} s_2 \\ \tfrac{1}{2} \text{ if } s_1 \sim_{\omega} s_2 \\ 0 \text{ if } s_2 \succ_{\omega} s_1 \end{cases} \quad (4.a.1)$$

at each $(s_1, s_2) \in S_1 \times S_2$ -- which, in the framework set out in Subsection 3.a), implies a similar equation for $\Pr^2_{\omega}(s_1, s_2)$. This assumption is commonly called "deterministic voting" (see, for instance, Mueller (2003: Chapter 11).

For the finite measure space of voters specified in Subsection 3.a), the simple majority rule relation can be defined as follows (as in McKelvey and Ordeshook (1976), Grandmont (1978) or McKelvey, Ordeshook and Ungar (1980)). First: Assume

$$\{\omega \in \Omega : x \succeq_{\omega} y\} \in \mathscr{F}(\Omega), \ \forall \ x, y \in X \quad (4.a.2)$$

Then the pairwise comparisons for the simple majority rule relation are: For each $x, y \in X$,

$$x \succeq^S y \text{ if and only if } \mu_{\Omega}\{\omega \in \Omega : x \succeq_{\omega} y\} \geq \mu_{\Omega}\{\omega \in \Omega : y \succeq_{\omega} x\} \quad (4.a.3)$$

As in Section 1: Saying that $x \in X$ is a "(weak) simple majority Condorcet winner" will mean $x \succeq^S y, \ \forall \ y \in X$; The asymmetric part of $\succeq^S$ will be denoted by $\succ^S$; Saying that $x \in X$ is a "(strong) simple majority Condorcet winner" will mean $x \succ^S y, \ \forall \ y \in X - \{x\}$.

4.b) Pure-strategy equilibria and Condorcet winners

When there is deterministic voting and the assumptions in Subsection 3.a) are satisfied, the objective functions based on votes that were discussed in Subsection 3.b) can be rewritten as functions of sets of voters with common preferences. For instance, a candidate's expected vote function can be written as

$$EV^c(s_1, s_2) = \mu_{\Omega}(\{\omega \in \Omega : s_c \succ_{\omega} s_k\}) + (\tfrac{1}{2})\cdot\mu_{\Omega}(\{\omega \in \Omega : s_c \sim_{\omega} s_k\}) \quad (4.b.1)$$

where k is the index for the "other candidate" (that is, k is the element in $C - \{c\}$). [The measurability of $\{\omega \in \Omega: s_c \succ_{\omega} s_k\}$and $\{\omega \in \Omega: s_c \sim_{\omega} s_k\}$ follows from (4.a.2) and the fact that $\mathscr{F}$ is a σ-field.]

Using (4.b.1), it is easy to see that, when the assumptions in Subsection 3.a) are satisfied *and* there is deterministic voting, the following two connections hold between the simple majority Condorcet winners for the relation defined by (4.a.3) and the pure-strategy equilibria for $(S_1, S_2; EV^1, EV^2)$: 1) (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if s_1^* and s_2^* are weak Condorcet winners; 2) If there is a strong Condorcet winner, then (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if $s_1^* = s_2^*$ *and* their common strategy (that is, $s \equiv s_1^* = s_2^*$) is the strong Condorcet winner.

For the game $(S_1, S_2; EV^1, EV^2)$, an "equilibrium strategy" for candidate c is an element of S_c that c chooses in at least one pure-strategy equilibrium this game (using, for instance, the definition of an equilibrium strategy for a player in a non-cooperative game in Vorobev (1977: p.3)). Since $(S_1, S_2; EV^1, EV^2)$ is constant-sum, (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if s_1^* is an "equilibrium strategy" for candidate 1 and s_2^* is an "equilibrium strategy" for candidate 2. So the result in the previous paragraph can be re-stated as: 1) For each $c \in C$, s_c^* is an equilibrium strategy for c if and only if s_c^* is a weak Condorcet winner; 2) When there is a strong Condorcet winner: For each $c \in C$, s_c^* is an equilibrium strategy for c if and only if s_c^* is the strong Condorcet winner.

Since the game $(S_1, S_2; EV^1, EV^2)$ is strategically equivalent to both $(S_1, S_2; VS^1, VS^2)$ and $(S_1, S_2; P\ell^1, P\ell^2)$, it follows that the results in the two previous paragraphs also hold for these two games.

These connections (and variations on them) have been discussed in Shubik (1968), Riker and Ordeshook (1973), Kramer (1977a, 1977b), McKelvey and Ordeshook (1990), Laffond, Laslier and Le Breton (1994), Laslier (1997), Ordeshook (1997) and other sources. The models of electoral competition discussed in Sections 5 and 6 will be ones where there is deterministic voting. In discussing those models, I will make use of these connections.

5. Unidimensional models with deterministic voting

The first unidimensional model that will be considered is the Hotelling-Downs model. In the formulation of the candidates' expectations about the voters' choices in Subsection 3.a), for any given c and (s_1, s_2), the expectations are a function of the voters' indexes. However, using the function m for the Hotelling-Downs model specified in Subsection 3.a), the expectations can be rewritten as a function of the voters' most-preferred alternatives.

Since the function m for the Hotelling-Downs model is a linear function of ω, it is invertible. Therefore (for any given c and (s_1, s_2)), for a voter who has s as his most-preferred alternative, the probability that he will vote for c is given by the $Pr^c_\omega(s_1, s_2)$ where $\omega = m^{-1}(s)$. This probability will be denoted by $Pr^c(s_1, s_2 | s)$. That is: For any given c and (s_1, s_2), $Pr^c(s_1, s_2 | s) \equiv Pr^c_\omega(s_1, s_2)$ at the s assigned to ω by m. $Pr^c(s_1, s_2 | s)$ gives us the candidates' expectations about the voters' choices as a function of the most-preferred alternatives. In what follows, an explicit equation for $Pr^c(s_1, s_2 | s)$ will be stated.

In Subsection 2.b), it was noted that (in the political reinterpretation of Hotelling's model), for any given $\omega \in \Omega$, $U_\omega(s) = -TC_\omega(s) = -(\kappa)\cdot|s - m(\omega)|$ will be a utility function on X. In addition, it was noted that each party will have the following expectations (for any given ω): *If* $TC_\omega(s_1) < TC_\omega(s_2)$, *then* the probability that ω will vote for party 1 is one; *If* $TC_\omega(s_1) > TC_\omega(s_2)$, *then* the probability that ω will vote for party 2 is one; *If* $TC_\omega(s_1) = TC_\omega(s_2)$, *then* the probability for each of ω 's two possible choices is one-half. Using the definition of the voter's total cost function, this implies that the parties have the following expectations: *If* $|s_1 - m(\omega)| < |s_2 - m(\omega)|$, *then* the probability that ω will vote for party 1 is one; *If* $|s_1 - m(\omega)| > |s_2 - m(\omega)|$, *then* the probability that ω will vote for party 2 is one; *If* $|s_1 - m(\omega)| = |s_2 - m(\omega)|$, *then* the probability for each of ω's two possible choices is one-half. That is: If the alternative selected by one party is closer (when measured with Euclidean distance) to a voter's most-preferred alternative, then the parties are certain he will vote for the closer party -- and, if the alternatives selected by the parties are equidistant from a voter's most-preferred ideology, then they consider his two possible choices to be equiprobable.

The conclusion reached at the end of the previous paragraph enables us to state the following explicit equation for the candidates' expectations about the voters' choices (as a function of the most-preferred alternatives). More specifically, for any given c and (s_1, s_2),

$$\Pr^c(s_1, s_2 | s) = \begin{cases} 1 \text{ if } |s_c - s| < |s_k - s| \\ \tfrac{1}{2} \text{ if } |s_c - s| = |s_k - s| \\ 0 \text{ if } |s_c - s| > |s_k - s| \end{cases} \tag{5.1}$$

(where k is the index for the other candidate).

I will illustrate how (5.1) can be applied, using the special case for the Hotelling-Downs model where X = [0, 100]. Using the distribution of most-preferred alternatives in the model,

$$EV^c(s_1, s_2) = \int_0^{100} \Pr^c(s_1, s_2 | s)ds \tag{5.2}$$

Using (5.1), we can solve the integral in (5.2) – thereby obtaining a different equation for $EV^c(s_1, s_2)$. In particular: Using the fact that $\Pr^c(s_1, s_2 | s) = \tfrac{1}{2}$, $\forall$ s for any particular (s_1, s_2) where $s_1 = s_2$ and using the function (of s) that is obtained from (5.1) for any particular (s_1, s_2) where $s_1 \neq s_2$, it follows that (for each $(s_1, s_2) \in S_1 \times S_2$)

$$EV^c(s_1, s_2) = \begin{cases} s_c + [(s_k - s_c) / 2] \text{ if } s_c < s_k \\ 50 \text{ if } s_c = s_k \\ [100 - s_c] + [(s_c - s_k) / 2] \text{ if } s_c > s_k \end{cases} \tag{5.3}$$

Therefore (using (5.3)), for each $(s_1, s_2) \in S_1 \times S_2$,

$$EV^c(s_1, s_2) = \begin{cases} (s_c + s_k)/2 & \text{if } s_c < s_k \\ 50 & \text{if } s_c = s_k \\ 100 - [(s_c + s_k)/2] & \text{if } s_c > s_k \end{cases} \tag{5.4}$$

An explicit equation like (5.4) can be used to learn some of the properties of a party's payoff function. For instance: (5.4) implies that, for any given $s_k' \in [0, 100]$,
1) when $s_c < s_k'$, party c's payoff is the value assigned by the linear equation

$$f_c(s_c) = [s_k' / 2] + (1/2)\cdot s_c \tag{5.5}$$

2) when $s_c > s_k'$, party c's payoff is the value assigned by the linear equation

$$g_c(s_c) = [100 - (s_k' / 2)] - (1/2)\cdot s_c \tag{5.6}$$

(5.4) also implies that, for any given $s_k' \neq 50$, there is a particular type of "discontinuity" in party c's payoff at $s_c = s_k'$. More specifically (for any given $s_k' \neq 50$): (i) the left-hand limit for party c's payoff at $s_c = s_k'$ is s_k', (ii) the right-hand limit for party c's payoff at $s_c = s_k'$ is $100 - s_k'$ and (iii) party c's payoff at $s_c = s_k'$ is 50 -- so these are three different numbers.

In the framework specified in Subsection 3.a), the Hotelling-Downs model corresponds to the supposition in the following theorem.

Theorem: Suppose $\Gamma = (S_1, S_2 ; EV^1, EV^2)$ satisfies the following assumptions: (i) $X = [a, b] \subset \mathbb{R}^1$, (ii) Γ is a regular model with a continuous uniform distribution of most-preferred alternatives on X, (iii) there is deterministic voting, and (iv) for each $\omega \in \Omega$, the preference ordering $\succeq_\omega$ satisfies

$$s_1 \succ_\omega s_2 \text{ if and only if } |s_1 - m(\omega)| < |s_2 - m(\omega)|$$

$$s_1 \sim_\omega s_2 \text{ if and only if } |s_1 - m(\omega)| = |s_2 - m(\omega)|$$

$$s_2 \succ_\omega s_1 \text{ if and only if } |s_1 - m(\omega)| > |s_2 - m(\omega)|$$

for each $(s_1, s_2) \in S_1 \times S_2$. Then x is an equilibrium strategy for a candidate if and only if x is the midpoint of the segment [a, b].

Since each candidate has a unique equilibrium strategy, a unique pure strategy equilibrium exists. Both Hotelling (1929) and Downs (1957) indicated a way in which each party might adjust its location in response to a location selected by the other party, and concluded that the parties "will converge on the same location" (Downs: p. 117). In the game-theoretic literature on electoral competition: For a given equilibrium, (s_1^*, s_2^*), saying that the parties "converge" at that equilibrium means that $s_1^* = s_2^*$ (see, for instance, McKelvey (1975: p. 820)). The theorem clearly implies that the parties converge in this sense. For other discussions of the Hotelling-Downs model, see (for instance) Shepsle and Bonchek (1997: pp. 104-109 in the section "Spatial Elections") and Aliprantis and Chakrabarti (2000: Example 2.11 on pp. 56-58).

Downs (1957) suggested the idea of considering other regular models with $X \subset \mathbb{R}^1$ where 1) "voters preferences are single-peaked ... [but] the slope downward from the apex need not be identical on both sides" (pp. 115-116) and 2) there is " a variable distribution of voters along the scale" (p. 117). Along these lines, various references have observed that the theorem stated above can be generalized to the following result (see, for instance, Davis and Hinich (1966: p.181), Osborne (1995: Proposition 1; 2004: Section 3.3], Aliprantis and Chakrabarti [2000: The "Generalized voter model" on pp. 66-67] or Roemer (2001: Theorem 1.1)).

Theorem: Suppose $\Gamma = (S_1, S_2; EV^1, EV^2)$ satisfies the following assumptions: (i) X is a convex subset of $\mathbb{R}^1$, (ii) Γ is a regular model with a continuous distribution of most-preferred alternatives on X, (iii) there is deterministic voting and (iv) for each $\omega \in \Omega$, the preference ordering $\succeq_\omega$ satisfies

$$[[z < y < m(\omega)] \text{ or } [z > y > m(\omega)]] \Rightarrow [y \succ_\omega z]$$

for each y, $z \in X$. Then x is an equilibrium strategy for a candidate if and only if x is the median for the distribution of most-preferred alternatives.

The first theorem established that, under the assumptions for the Hotelling-Downs model, the candidates' equilibrium strategy is the midpoint of X. Under the assumptions for that model, it is also the median and the mean for the distribution of most-preferred alternatives. So one will conclude that the social choice made through the electoral competition is "centrist" if any of these three measures of center is used. Under the assumptions for the second (more general) theorem, the candidates' equilibrium strategy continues to be the median for the distribution of most-preferred alternatives, but it won't be the midpoint of X unless the midpoint happens to coincide with the median. Similarly, it won't be the mean for the distribution of most-preferred alternatives unless the mean happens to coincide with the median. So, under the assumptions for the more general theorem, *in most cases* the candidates' equilibrium strategy will be neither the midpoint of X nor the mean for the distribution of most-preferred alternatives. Therefore *in most cases* one's conclusion about whether the social choice is "centrist" will depend on the measure of center that is used.

Analogous results have also been obtained for models where the set of voters is finite. For instance, using the connection between Condorcet winners and equilibrium strategies discussed in Subsection 4.c), the first theorem stated in Section 1 implies the following result.

Theorem: Suppose $\Gamma = (S_1, S_2; EV^1, EV^2)$ satisfies the following assumptions: (i) the set of voters is finite, (ii) $(X, (\succeq_1,\ldots,\succeq_{\#\Omega}))$ is a regular model, (ii) there is an ordering relation $\leq_o$ which is such that, for each $\omega \in \Omega$, the preference ordering $\succeq_\omega$ is single-peaked with respect to $\leq_o$, and (iii) there is deterministic voting. Then $x \in X$ is an equilibrium strategy for a candidate if and only if x is a median for the distribution of most-preferred alternatives (*with respect to* $\leq_o$).

This theorem clearly implies: 1) If there is an *odd* number of voters, then (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if $s_1^* = s_2^*$ and s_1^* is the unique median for the distribution of most-preferred alternatives; 2) If there is an *even* number of voters, then (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if s_1^* and s_2^* are medians for the distribution of most-preferred alternatives.

These conclusions about electoral competition models are based on Black's analysis of committees. One of the first references to explicitly state *implications* from Black's analysis of committees *for* electoral competition models with a finite number of voters is Barr and Davis (1966: Theorem 1). The implications for electoral competition models with a finite number of voters that follow from Black's analysis of committees have been discussed in Ordeshook (1986: Theorem 4.3 on p. 162; 1992: pp. 103-105), Banks (1991: p. 58), Bierman and Fernandez (1998: Section 5.3), Brams (2003: pp. 598-602) and other references.

These conclusions about electoral competition models with a finite number of voters have some noteworthy features. For instance, in these models, a pure-strategy equilibrium exists. So these theorems imply there are certain policies which each candidate can be expected to embody. In addition, each candidate selects a strategy which is a median for the distribution of most-preferred alternatives.

The specific equilibrium strategies for the candidates establish that the policies a candidate can be expected to embody will be at the median for the distribution of most-preferred alternatives. What's more, when there is an *odd* number of voters, the policies (which the candidates can be expected to embody) are unique and will converge. However, when there is an *even* number of voters, it's possible to have more than one median for the distribution of most-preferred alternatives. So, when there is an *even* number of voters, the policies (which the candidates can be expected to embody) (1) are not unique and (2) could converge, but won't necessarily converge.

Kramer (1973, 1976) established that, when X is a subset of a Euclidean space, assuming single-peakedness is tantamount to requiring X to be one-dimensional. The next section will consider models where X is a subset of a Euclidean space, but is not limited to being one-dimensional.

6. Finite-dimensional models with deterministic voting

Black and Newing (1951) and Black (1958: pp. 131-137) considered a model which is closely related to the committee model discussed in Section 1 -- with the only significant change being that these references *replaced* the assumption of a unidimensional set of motions *with* the assumption that each motion either has two distinguishable aspects or consists of two separate parts. These references pointed out that, in these circumstances, a motion would be defined by two characteristics - - and used a real-valued variable for each characteristic.

Significantly, they showed that, for their two-dimensional model, the assumption that the voters have strictly quasi-concave utility functions on the set of motions is *not* sufficient to assure that a Condorcet winner exists. Using the connection between Condorcet winners and pure-strategy equilibria discussed at the end of Section 4, it follows that, when S is multidimensional and voting is deterministic, assuming that the voters have strictly quasi-concave utility functions on the set of motions is *not* sufficient to assure that a pure-strategy equilibrium exists in an electoral competition.

6.a) Plott's analysis

As in the earlier work by Black and Newing (1951), research by Plott (1967) analyzed a model of a committee. He assumed that the committee is attempting to decide on the magnitude of several variables (p. 787) (more precisely: there is a set of alternatives, which is a subset of $\mathbb{R}^h$ -- where h is an integer greater than 1). He assumed that each voter has a differentiable utility function, U_ω, on the set of possible alternatives (p. 788).

Plott started with the following "notion of equilibrium":

> "...the variables could be changed by any amount. If a change in the variables is proposed and the change does not receive a majority vote, then the 'existing state' of the variables remains. If no possible change in the variables could receive a majority vote, then the 'existing state' of the variables is an equilibrium" (p. 787)

Plott noted that an alternative is a "global" equilibrium of this sort (where changes of any amount are considered) only if it is also a "local" equilibrium. In his analysis, Plott explicity considered only local equilibria.

When studying local equilibria, Plott specifically considered "motions", $dx = (dx_1,...,dx_h)$ from an "existing situation," x. In the text of his article (see p. 788), Plott suggested that $(dx_1,...,dx_h)$ could be taken to be a "small change" in x. With this interpretation, the inner product $\langle dx, \nabla U_\omega(x)\rangle$ is a total differential. He also suggested that (in footnote 4 on p. 788) $(dx_1,...,dx_h)$ could be taken to be a direction. With this interpretation, $\langle dx, \nabla U_\omega(x)\rangle$ is the directional derivative of U_ω at x (for the direction dx).

Plott considered a committee with an odd number of voters (p. 790). Plott started with a setting in which there are no constraints on the set of alternatives. Plott assumed an individual "votes for" a motion dx <u>if</u> $\langle dx, \nabla U_\omega(x)\rangle > 0$ (p. 788). In addition, he considered settings in which an individual also "votes for" a motion <u>only if</u> $\langle dx, \nabla U_\omega(x)\rangle > 0$ (see part A of Section II, on p. 790). In these settings, saying that an alternative x is a local equilibrium means that, at x, no motion could receive a majority vote. That is, there is no motion dx where $\langle dx, \nabla U_\omega(x)\rangle > 0$ for more than / 2) voters.

One of the things that Plott proved (see p. 790 and Theorem 1 on p. 797) is: For an alternative, x, to be a local equilibrium, there must be at least one voter, ω, with $\nabla U_\omega(x) = \underline{0}$. Then Plott considered an alternative, x, which is "a maximum for one and only one individual" (p. 790). And he derived the following necessary condition for such an alternative to be a local equilibrium (see p. 790 and Theorem 2 on p. 799): The remaining individuals can be divided into pairs in such a way that, for each pair $\{\alpha, \beta\}$, there exist positive real numbers y_α, y_β such that

$$y_\alpha \cdot \nabla U_\alpha(x) \;+\; y_\beta \cdot \nabla U_\beta(x) \;=\; \underline{0} \qquad (6.a.1)$$

Letting $\gamma(\alpha, \beta) = y_\beta / y_\alpha$, (6.a.1) implies

$$\nabla U_\alpha(x) \;=\; -\gamma(\alpha, \beta) \cdot \nabla U_\beta(x) \qquad (6.a.2)$$

As Plott points out, this tells us that "all individuals for which the point is not a maximum can be divided into pairs whose interests are diametrically opposed" (p. 790). Plott also obtained analogous results for settings where there is "a single constraint such as a budget constraint" (p. 792).

Plott's results led him to conclude: "The most important point is that there is certainly nothing inherent in utility theory which would assure the existence of an equilibrium. In fact, it would only be an accident (and a highly improbable one) if an equilibrium exists at all" (pp. 790-792).

Plott's results were specifically for local equilibria. However, Sloss (1973) subsequently established that Plott's conditions are necessary and sufficient for the existence of a global equilibrium when Plott's original assumptions are supplemented with the following two assumptions: 1) X is an open set and 2) each voter's preferences on X can be represented by a pseudo-concave utility function.

When the results discussed in this Subsection are combined with the connection between Condorcet winners and pure-strategy equilibria discussed at the end of Section 4, we get the following analogous conclusion for electoral competition models with finite sets of voters and deterministic voting when S is multidimensional: While pure-strategy equilibria sometimes exist in these electoral competitions, the existence of a pure-strategy equilibrium is something which is relatively rare.

6.b) Sufficient conditions

In the results discussed in Subsection 6.a), Plott established important *necessary* conditions for the type of equilibrium that he analyzed. More specifically, those results can be rephrased as follows: 1) Under the assumptions stated in Subsection 6.a), a *necessary* condition for x being the type of equilibrium that Plott analyzed is: At least one voter has a zero utility gradient at x; 2) When one also assumes there is exactly one voter with a zero utility gradient at x, a *necessary* condition for x being the type of equilibrium that Plott analyzed is: The other voters can be divided into pairs where the utility gradient (at x) for one of the voters in the pair is a negative scalar multiple of the utility gradient (at x) for the other voter in the pair.

Other authors have identified *sufficient* conditions for equilibria in multidimensional models. Some of the pioneering work on this topic was done by Tullock. In Tullock (1967a) and in Chapter III of Tullock (1967b), he considered the simple majority rule relation in a setting where the set of alternatives has two dimensions -- and identified *sufficent* conditions for the existence of a strong Condorcet winner. In Chapter IV of Tullock (1967b), he extended the Hotelling-Downs model to a setting where the strategy set for the candidates is an issue space which has two dimensions. What's more, in Chapter IV, he pointed out that his *sufficent* conditions for the existence of a strong Condorcet winner were also *sufficent* conditions for the existence of a pure-strategy equilibrium in an electoral competition model.

I will first discuss Tullock's conditions in the context of the simple majority rule relation. After stating implications of his conditions for strong Condorcet winners, I will then discuss Tullock's conditions in the context of electoral competitions.

In the context of the simple majority rule relation, there are three key parts for Tullock's conditions. One part is: There is a set of social alternatives which is a rectangular region in a plane. The second part is: Each voter has a most-preferred alternative and the most-preferred alternatives are uniformly distributed on the rectangular region. The third part (of Tullock's conditions) is: For each voter, one alternative (in the rectangular region) is at least as good as another alternative *if and only if* the Euclidean distance from the alternative to his most-preferred alternative is less than or equal to the Euclidean distance from the other alternative to his most-preferred alternative.

The reasoning that Tullock used establishes that, under these conditions, a social alternative is a strong Condorcet winner if and only if it is the the center of the rectangular region. So Tullock's conditions are *sufficent* conditions for the existence of a strong Condorcet winner.

In the extension of the Hotelling-Downs model that is in Chapter 4 of Tullock (1967b), he assumed that the set of possible strategies for the candidates is the set of social alternatives considered above. In other, words, a candidate's strategy is a choice of a social alternative. This assumption can be interpreted within the framework in Section 3 as: $X \subseteq \mathbb{R}^2$ and X is a rectangular region.

Within the framework in Section 3, the second part of Tullock's conditions can be interpreted as: Γ is a regular model, with a uniform distribution of most-preferred alternatives.

The assumption about a voter's preferences in the third part (of Tullock's conditions) is similar to the following assumptions in Hotelling's locational model for duopolists: Each buyer will transport his purchases home at a constant per unit cost and each buyer's objective is to buy his "unit quantity" at a store which minimizes his transportation cost. In a model of electoral competition, this becomes the assumption that (for any given voter) one location is at least as good as another location *if and only if* the Euclidean distance from the location to the voter's most-preferred alternative is less than or equal to the Euclidean distance from the other location to the voter's most-preferred alternative. The specific assumption that Tullock used is stated precisely in the next paragraph.

As has been noted, Tullock considered a set of alternatives which is in $\mathbb{R}^2$ and is a rectangular region. Take the vectors in $\mathbb{R}^2$ to be column vectors. For any $v \in \mathbb{R}^2$, let v^t denote the transpose of v. Let I_2 denote the (2x2) identity matrix. Euclidean distance for $\mathbb{R}^2$ is (of course) derived from the Euclidean norm - - which, for $\mathbb{R}^2$, is $\|y\|_2 = (y^t I_2 y)^{1/2}$.

For any given , the assumption "For , one alternative in X is at least as good as another alternative if and only if the Euclidean distance from the alternative to his most-preferred alternative is less than or equal to the Euclidean distance from the other alternative to his most-preferred alternative" can be stated precisely as:

$$x \succ_\omega y \text{ if and only if } \|x - m(\ )\|_2 < \|y - m(\ )\|_2 \qquad (6.b.1)$$

and

$$x \sim_\omega y \text{ if and only if } \|x - m(\ )\|_2 = \|y - m(\ )\|_2 \qquad (6.b.2)$$

for each $x, y \in S$.

For each , the corresponding preferences can be represented by the following ordinal utility function on X:

$$U_\omega(x) = -[x - m(\omega)]^t I_2 [x - m(\omega)] \qquad (6.b.3)$$

The expression on the right-hand side of (6.b.3) is (of course) a specific quadratic form. Since this particular quadratic form is being used: If the domain of the function in (6.b.3) was all of $\mathbb{R}^2$, then the graph of the function in x_1-x_2-U_ω space would be a circular paraboloid with a global maximum at $m(\omega)$. The graph of the function in (6.b.3) is therefore the portion of that paraboloid which is above the rectangular region X. The global maximum for the utility function is (of course) 0. Hence, for each $x \in X - \{m(\omega)\}$, we have $U_\omega(x) < 0$. For each possible $k < 0$, (6.b.3) implies that the corresponding indifference curve, $\{x \in X: U_\omega(x) = k\}$, is the *intersection of* a circle (where the length of the radius is the positive root of -k and where the center is the voter's most-preferred alternative) *and* the rectangular region X. So these indifference curves are the portions of the concentric circles centered at $m(\omega)$ which are in X. For any two elements in $X - \{m(\omega)\}$ which are on different indifference curves, the voter will prefer the alternative on the indifference curve for which the corresponding circle has the smaller radius.

In Tullock's extension of the Hotelling-Downs model, it is assumed that each voter learns which social alternative is chosen by each candidate. It is also assumed that each voter evaluates the candidates entirely in terms of the social alternatives the candidates have chosen. Within the framework summarized in Section 3, these assumptions can be interpreted as: There is deterministic voting, with each $\omega \in \Omega$ having a preference ordering R_ω on X which satisfies (6.b.1) and (6.b.2) for each x, y $\in$ X.

Within the framework in Section 3, Tullock's result for his extension of the Hotelling-Downs model can be stated as follows.

Theorem: Suppose $\Gamma = (S_1, S_2; EV^1, EV^2)$ satisfies the following assumptions: (i) $X \subseteq \mathbb{R}^2$ and X is a rectangular region, (ii) Γ is a regular model, with a uniform distribution of most-preferred alternatives, and (iii) Γ has deterministic voting, with each $\omega \in \Omega$ having a preference ordering, R_ω, on X, which satifies (6.b.1) and (6.b.2) for each x, y $\in$ X. Then (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if $s_1^* = s_2^*$ and their common strategy is the center of the rectangular region.

This tells us that, when a model of electoral competition satisfies Tullock's conditions, some of the important implications that follow are similar to ones that hold for the unidimensional model developed by Hotelling and Downs. For one thing: Tullock's conditions imply the existence of a unique pure-strategy equilibrium. For another: Tullock's conditions imply convergence for the candidates' strategies. In addition, under Tullock's conditions, the common strategy for the candidates is in a "central location".

In some references, 1) the assumption about the set of alternatives has been generalized so that $X \subseteq \mathbb{R}^h$ (where h is a positive integer) *and* 2) the assumptions about voter preferences stated near the beginning of this Subsection have been generalized to let them be based on any metric in a certain class of metrics (which has the Euclidean metric as one of its elements). The second generalization has been accomplished as follows. Take the vectors in $\mathbb{R}^h$ to be column vectors. For any $v \in \mathbb{R}^h$, let v^t denote the transpose of v. Euclidean distance for $\mathbb{R}^h$ (of course) is derived from the Euclidean norm for $\mathbb{R}^h$, which is $\|y\|_h = [y^t I_h y]^{½}$ (where I_h denotes the (hxh) identity matrix). For a given h, let A denote a symmetric, positive-definite, (hxh) matrix. The norm $\|y\|_A = [y^t A_h y]^{½}$ is called the "weighted Euclidean norm". For any x,y $\in$ X, the number $\|x - y\|_A$ is called the "weighted Euclidean distance" between x and y. Voter preferences are based on weighted Euclidean distance by assuming that, for any given $\omega \in \Omega$,

$$x \succ_\omega y \text{ if and only if } \|x - m(\;)\|_A < \|y - m(\;)\|_A \tag{6.b.4}$$

and

$$x \sim_\omega y \text{ if and only if } \|x - m(\;)\|_A = \|y - m(\;)\|_A \tag{6.b.5}$$

for each x, y $\in$ X. That is: For any given voter, one alternative in X is at least as good as another alternative if and only if the weighted Euclidean distance from the alternative to his most-preferred alternative is less than or equal to the weighted Euclidean distance from the other alternative to his most-preferred alternative.

When this assumption is made: For each $\omega \in \Omega$, the corresponding preferences can be represented by the ordinal utility function

$$U_\omega(x) \;=\; -[x - m(\omega)]^t\, A\, [x - m(\omega)] \tag{6.b.6}$$

on X.

As in (6.b.3), the expression on the right-hand side of (6.b.6) is a quadratic form. This time the form uses the matrix A. Since this particular quadratic form is being used: If $X = \mathbb{R}^2$, then the graph of (6.b.6) in x_1-x_2-U_ω space is an elliptical paraboloid with a global maximum at $m(\omega)$. This implies a voter's indifference curves would be ellipses which are concentric (with the common center for the ellipses being his most-preferred alternative). When (as in Tullock's analysis) we have $X \subset \mathbb{R}^2$, a voter's indifference curves would be the portions of these ellipses which are in X. That is, each indifference curve would be the *intersection of* one of the ellipses *and* X. In either case (that is, if either $X = \mathbb{R}^2$ or $X \subset \mathbb{R}^2$): For any two alternatives that are not on the same indifference curve, the voter prefers the alternative whose indifference curve is closer (in weighted Euclidean distance) to his most-preferred alternative

Davis and Hinich (1966) specified a model of electoral competition and used assumptions like these as one part of a set of *sufficient* conditions for the existence of a pure-strategy equilibrium. Their conditions correspond to the premise in the following theorem.

Theorem: Suppose $\Gamma = (S_1, S_2; EV^1, EV^2)$ satisfies the following assumptions: (i) $X = \mathbb{R}^h$, (ii) Γ is a regular model, with a multivariate normal distribution of most-preferred alternatives, and (iii) Γ has deterministic voting where, for each $\omega \in \Omega$, (6.b.4) and (6.b.5) hold for each x, y $\in$ X. Then (s_1^*, s_2^*) is a pure-strategy equilibrum if and only if $s_1^* = s_2^*$ and their common strategy is the mean for the distribution of most-preferred alternatives.

Among other things, this theorem establishes that (like Tullock's conditions) Davis and Hinich's conditions imply 1) there is a unique pure-strategy equilibrum, 2) the candidates' strategies converge and 3) the candidates' strategies are in a "central location".

There are also other conditions for regular models with infinite sets of voters which have been shown to imply the existence of a Condorcet winner (and, hence, also pure-strategy equilibria in electoral competitions with deterministic voting) when there is a multimensional X. Davis and Hinich (1968, 1971) obtained results for other models with multivariate normal distributions. Davis, Hinich and Ordeshook (1970), Riker and Ordeshook (1973) and McKelvey (1975) analyzed multivariate distributions that are symmetric. Davis, DeGroot and Hinich (1972) and

Hoyer and Meyer (1975) worked with multivariate distributions which have a total median. McKelvey, Ordeshook and Ungar (1980) studied multivariate distributions that are weakly symmetric. In addition, extensions of Tullock's model which do not require each voter to have a unique most-preferred alternative were developed by Arrow (1969), Grandmont (1978) and others.

Conditions for models with finite sets of voters which imply the existence of a Condorcet winner (and, hence, also pure-strategy equilibria in electoral competitions with deterministic voting) when there is a multidimensional set of conditions have also been identified. Important results along these lines are in Simpson (1969), Sloss (1973), Wendell and Thorson (1974), McKelvey and Wendell (1976), Matthews (1979, 1980), Enelow and Hinich (1983), Saari (1997) and other sources.

Significantly, in this work, the sufficient conditions either have demanding symmetry requirements for the distribution of most-preferred alternatives or have other demanding requirements -- e.g., the convexity assumption used by Arrow (1969), the symmetry assumption for sets of intermediate preferences used by Grandmont (1978) or the assumptions about the number of voters whose most-preferred alternative is located at the equilibrium that are in Slutsky (1979) and Enelow and Hinich (1983).

For further discussion about conditions which are sufficient for Condorcet winners when there is a multidimensional alternative set (and pure-strategy equilibria in electoral competition models with deterministic voting and multidimensional strategy sets), see (for instance) Feld and Grofman (1987), Straffin (1989, 1994), McKelvey (1990: Section 3), Owen (1995: Section XVI.1), Mas-Colell, Whinston and Green (1995: p. 805-806), Hinich and Munger (1997: pp. 87-88) or Saari (2004).

6.c) McKelvey's theorem

McKelvey (1976) discovered an important result about majority rule. He also identified some implications for models of committees.

In his analysis, McKelvey assumed a particular type of majority rule which is different from the "simple majority rule" used in the discussion about Black's analysis of committees in Section 1. The particular type he assumed requires an absolute majority (that is, more than half of the entire set of voters) for majority preference. More specifically: (i) one alternative beats another one if and only if an absolute majority prefer it and (ii) otherwise, the two alternatives tie one another.

McKelvey (1976) considered settings where there is a finite set of voters with ≥ 3. Recall from Subsection 1.d): For any given $(X, (\succeq_1,\ldots,\succeq_{\#\Omega}))$, the relation "beats or ties" for absolute majority rule is denoted by $\succeq^A$; For any given $(X, (\succeq_1,\ldots,\succeq_{\#\Omega}))$, saying that $x \in X$ is a "(weak) absolute majority Condorcet winner" (or, equivalently, is in the core for a committee that is using absolute majority rule) means $x \succeq^A y, \forall\, y \in X$.

McKelvey (1976) considered settings where (1) X is a multidimensional Euclidean space and (2) voters have preferences like the ones used in Tullock's conditions. Tullock (1967a, 1967b), of course, only considered a setting where $X \subseteq \mathbb{R}^2$ and X is a rectangular region. For settings where X is any multidimensional Euclidean space and voters have preferences like the ones used in Tullock's conditions: Suppose 1) For an integer $h \geq 2$, $X = \mathbb{R}^h$ and 2) For each voter, one element in X is at least as good as another element if and only if the Euclidean distance from the element to his most-preferred element is less than or equal to the Euclidean distance from the other element to his most-preferred element.

The second assumption stated above can be stated precisely by using the Euclidean norm for $\mathbb{R}^h$ to generalize (as follows) the assumptions about preferences stated in (6.b.1) and (6.b.2) -- so that the generalized assumptions apply to $X = \mathbb{R}^h$. For any given $\omega \in \Omega$,

$$x \succ_\omega y \text{ if and only if } \|x - m(\ \ )\|_h < \|y - m(\ \ )\|_h \tag{6.c.1}$$

and

$$x \sim_\omega y \text{ if and only if } \|x - m(\ \ )\|_h < \|y - m(\ \ )\|_h \tag{6.c.2}$$

for each $x, y \in X$. When $X = \mathbb{R}^h$, a voter whose preferences satisfy this assumption is said to have "Euclidean preferences".

McKelvey proved the following theorem.

Theorem: Suppose $(X, (\succeq_1, \ldots, \succeq_{\#\Omega}))$ is a regular model where ≥ 3, $X = \mathbb{R}^h$ (with $h \geq 2$) and each voter has Euclidean preferences. If there is no (weak) absolute majority Condorcet winner, then for any $x, y \in X$ there exists a finite sequence of alternatives $\{z_1, \ldots, z_N\}$ with $z_1 = x$ and $z_N = y$ such that $z_{j+1} \succ^A z_j$, $\forall j \in \{1, \ldots, N-1\}$

McKelvey's analysis used absolute majority rule. However, when one makes the other assumptions that he used, an alternative is a (weak) absolute majority Condorcet winner if and only if it is a (weak) simple majority Condorcet winner (see, for instance, Theorem 1.1 on p. 146 in McKelvey and Wendell (1976) or Proposition 1 on p. 318 in Kramer (1977b)). So his result applies to both types of (weak) Condorcet winner.

The theorem tells us that the following property holds under McKelvey's assumptions when there is no (weak) Condorcet winner: For any given pair of alternatives, there is a sequence (1) that starts at the first alternative in the pair and ends at the second one in the pair and (2) where each alternative in the sequence is preferred by an absolute majority of voters to the one that immediately precedes it. So when there is no (weak) Condorcet winner, it is possible for such sequences to wander all over the set of alternatives.

6.d) Summing up

The electoral competition models with deterministic voting and unidimensional strategy sets based on the work of Hotelling and Downs and the work of Black have pure-strategy equilibria -- and the candidate strategies in those equilibria are at central locations. In addition, the research discussed in Subsections 6.a) and 6.b) identified conditions under which there are pure-strategy equilibria in electoral competition models with deterministic voting and multidimensional strategy sets -- and the candidate strategies in those equilibria are at central locations. But, significantly, the sufficient conditions for pure-strategy equilibria in the latter case (that is, for pure-strategy equilibria in electoral competition models with deterministic voting and multidimensional strategy sets) are severe conditions that are unlikely to be satisfied.

7. Probabilistic voting models

7.a) An overview

The results for electoral competition models discussed in the previous Sections were for models with deterministic voting. That is, they were for models where the alternatives selected by the candidates always fully determined the choice that every non-indifferent voters would make. More specifically, whenever a voter preferred the alternative selected by one candidate to the alternative selected by the other candidate , that voter was certain to vote for the candidate with the preferred alternative.

Subsection 7.b) and Sections 8-10 are about electoral competition models which use a probabilistic model of voter choice – that is, one that predicts the probabilities that a voter choose each. In other words, they will be about models which do *not* assume that the candidates are certain about the choices that will be made by all of the non-indifferent voters. Such models are commonly called "probabilistic voting models"-- reflecting the fact that the candidates' uncertainty can be modeled by using a probabilistic description of the voters' choice behavior. Subsection 7.b) will discuss the reasons why people have been interested in probabilistic voting models. The next two Sections will concentrate on what has been learned about the implications of candidate uncertainty for the existence and location of a pure-strategy equilibrium.

7.b) Reasons for analyzing probabilistic voting models

Various researchers have become interested in the implications of candidate uncertainty about voters' choices primarily because there are good empirical reasons for believing that actual candidates often *are* uncertain about the choices that voters are going to make on election day. First, candidates tend to rely on polls for information about how voters will vote, but "information from public opinion surveys is not error-free and is best represented as statistical" (Ordeshook (1986: p. 179)). Second, even when economists and political scientists have developed sophisticated statistical models of voters' choices and have used appropriate data sets to estimate them, there has consistently been a residual amount of unexplained variation (see, for instance, Fiorina (1981); Enelow and Hinich (1984: Chapter 9); Enelow, Hinich, and Mendell (1986); Merrill and Grofman (1999)).

These circumstances have led many empirically-oriented public choice scholars to the following view, expressed in Fiorina's empirical analysis of voting behavior: "In the real world choices are seldom so clean as those suggested by formal decision theory. Thus real decision makers are best analyzed in probabilistic rather than deterministic terms" (1981: p. 155). Some theoretically-oriented public choice scholars have also adopted the same view and have developed and analyzed the theoretical properties of models in which candidates are assumed to have probabilistic (rather than deterministic) expectations about voters' choices. More specifically, these theorists have carried out these studies because, as Ordeshook put it, "if we want to design models that take cognizance of the kind of data that the candidates are likely to possess, probabilistic models seem more reasonable" (1986: p. 179).

In the context of models where X is a set of possible positions on an issue, when the assumption of deterministic voting is used, each non-indifferent voter will definitely vote for the candidate whose position is preferred by that voter. In an analysis of a model of this sort (where they also assumed that, for each voter, there is a utility function on the set of possible positions *and* the magnitudes of utility differences are meaningful), Merrill and Grofman (1999: p. 81) have argued

> "Yet voters use criteria other than issues to choose candidates. The probability, furthermore, that even an issue-oriented voter will select the candidate of higher utility is certainly less that unity if utilities do not differ greatly."

Merrill and Grofman (1999: p. 82) have also pointed out that one can "include non-issue effects by adding a probabilistic (or stochastic) component to the issue-oriented utility". This method for including non-issue effects is one (influential) way of specifying what is sometimes called a "probabilistic model of voter choice".

It is reasonable to take the view that deterministic voting models are most appropriate for elections with candidates who are well informed about the voters and their preferences. However, it is also reasonable to think that probabilistic voting models are most appropriate for elections in which candidates have incomplete information about voters' preferences and/or there are random factors that can potentially affect voters' decisions. Because most elections are in this second category, it seems appropriate to agree with Calvert (1986) that assuming "that candidates cannot perfectly predict the response of the electorate to their platforms is appealing for its realism" (p. 14), a conclusion that he points out, is in harmony with the "importance attached by traditional political scientists to the role of imperfect information in politics" (p. 54).

8. Unidimensional models with probabilistic voting

Comaner (1976) and Hinich (1977) did pioneering research on unidimensional models where candidates are uncertain about whom the individual voters in the electorate will vote for. In their articles, they independently re-examined models with single-peaked preferences and addressed the question: Is choosing a median necessarily an equilibrium strategy?

Comaner (1976) and Hinich (1977) showed that, in models with single-peaked preferences, <u>if</u> there is indeterminateness in voter choices <u>then</u> choosing a median might not be an equilibrium strategy. This result will initially be illustrated with the following example (which is similar to an example used in Hinich (1977: p. 213) to establish the result). The authors were concerned with the consequences of indeterminatess being introduced into the type of electoral competition model that was discussed in the previous sections. As in Hinich's original example, non-deterministic voting will be assumed for the choices of some (but not all) voters in the initial illustration in this Section.

Example 8.1: Let $= \{1, 2, 3\}$. Assume $X = [-1,+1]$. Assume the voters' most-preferred alternatives are $m(1) = -1$, $m(2) = 0$ and $m(3) = +1$. Assume that, for each , there is a difference-scale utility function:

$$U_\omega(x) = - \ x - \ m(\ ) \tag{8.1}$$

Also assume that, for each , there is a function P_ω (whose domain contains the set $\{z \in \mathbb{R} \mid \exists$ $(s_1, s_2 \in X \times X$ such that $U_\omega(s_1) - U_\omega(s_2) = z\}$ and whose range is contained in the set [0,1]) such that, for each $(s_1, s_2 \quad X \times X$,

$$Pr^1{}_\omega(s_1, s_2) = P_\omega(U_\omega(s_1) - U_\omega(s_2)) \tag{8.2}$$

By (8.2) and (8.1),

$$Pr^1{}_\omega(s_1, s_2) = P_\omega(|s_2 - m(\ )| - |s_1 - m(\ )|) \tag{8.3}$$

Assume that $P_1(y) = P_2(y)$, y and that (for $= 1,2$) (a) P_ω is differentiable, (b) $P_\omega^{/}(y) \quad 0$, y, (c) there exists an $r > 0$ such that $P_\omega^{/}(y) > 0$, y $[0,r)$ and (d) $P_\omega(0) = ½$.

For $\omega = 3$, assume deterministic voting. Using (8.2), it follows that

$$P_3(y) = \begin{cases} 1 \text{ if } y > 0 \\ \frac{1}{2} \text{ if } y = 0 \\ 0 \text{ if } y < 0. \end{cases} \tag{8.4}$$

The following argument shows that, even though the median for the distribution of most-preferred alternatives, $x_{med} = 0$, is a feasible strategy for each candidate, (x_{med}, x_{med}) is *not* a pure-strategy equilibrium in the two-candidate game $(S_1, S_2; EV^1, EV^2)$. The assumptions in the example imply there exists some (0,r) such that $Pr^1_1(\ , x_{med}) >$ $Pr^1_2(\ , x_{med}) > 1/4$ and $Pr^1_3(\ , x_{med}) = 1$. This implies $EV^1(\ , x_{med}) > 3/2$. Since $EV^1(x_{med}, x_{med}) = 3/2$, candidate 1 would be better off if he unilaterally changed his strategy from x_{med} to . Therefore this example shows that a median is not necessarily an equilibrium strategy for a candidate when probabilistic voting is introduced.

The reason for including Example 8.1 was to illustrate the type of example used in Hinich (1977). The analysis of this example (in the preceding paragraph) also illustrated the type of logical argument that he used.

Hinich's original example assumed deterministic voting for one voter. Since that assumption was in his example, it was also included in Example 8.1. Because there is a deterministic voter in Hinich's example and Example 8.1, they clearly establish that having some deterministic voting (in a model with single-peaked preferences) is NOT sufficient for a median to be an equilibrium strategy.

At the same time, because there is a discontinuity at $y = 0$ for the P_ω of the deterministic voter in both Example 8.1 and Hinich's example, those examples don't establish whether (in a model with single-peaked preferences) a median can fail to be an equilibrium strategy IF each P_ω is assumed to be continuous. A fortiori, they also don't settle this question for the stronger assumption of differentiability for each P_ω.

The following variation on Example 8.1 will illustrate the fact that the same basic reasoning can be used without assuming there is a non-differentiable (or, alternatively, a discontinuous) P_ω.

Example 8.2: Everything assumed in Example 8.1 up through (8.3) will also be assumed in this example. However, unlike in Example 8.1, in this example there will be a specific (and differentiable) P_ω function for *each* voter.

For each , we will use the following function (with the domain $S_1 \times S_2$)

$$z_\omega(s_1, s_2) = |s_2 - m(\omega)| - |s_1 - m(\omega)| \tag{8.5}$$

For each , the domain of P_ω will be an open interval which contains the range of the function z_ω.

For = 1 and = 2, assume that (at each element in its domain)

$$P_{\infty}(y) = \begin{cases} 1 \text{ if } y \geq 1 \\ \tfrac{1}{2} + (\tfrac{3}{4})y - (\tfrac{1}{4})(y^3) \text{ if } -1 < y < +1 \\ 0 \text{ if } y \leq -1 \end{cases} \tag{8.6}$$

For = 3, assume that (at each element in its domain)

$$P_3(y) = \begin{cases} 1 \text{ if } y \geq \tfrac{1}{6} \\ \tfrac{1}{2} + (\tfrac{3}{4})y - (\tfrac{1}{4})(y^3) \text{ if } -\tfrac{1}{6} < y < +\tfrac{1}{6} \\ 0 \text{ if } y \leq -\tfrac{1}{6} \end{cases} \tag{8.7}$$

Suppose $s_1 = 1/3$ and $s_2 = x_{med}$. Then we have $EV^1(1/3, x_{med}) = P_1(z_1(1/3,0)) + P_2(z_2(1/3,0)) + P_3(z_3(1/3,0)) \approx .2595 + .2595 + 1$. Therefore $EV^1(1/3, x_{med}) \approx 1.518$. Since $EV^1(x_{med}, x_{med}) = 1.5$, it follows that (x_{med}, x_{med}) is <u>not</u> a pure strategy equilibrium in $(S_1, S_2; EV^1, EV^2)$.

Comaner (1976) provided examples with skewed distributions of most-preferred alternatives where there is a pure-strategy equilibrium at an alternative that is not a median. Hinich (1977) provided two examples where there is a pure-strategy equilibrium at an alternative measure of the center for the distribution of most-preferred alternatives. The first example illustrated the fact that, when the candidates are uncertain about the voters' choices, there can be a pure-strategy equilibrium at the mean for the distribution of most-preferred alternatives, rather than at the median. It also showed that the equilibrium can be far from the median. The second of these examples illustrated the fact that, when the candidates are uncertain about voters' choices, there can be a pure-strategy equilibrium at the mode for the distribution of most-preferred alternatives, rather than at either the median or the mean. Related analyses of unidimensional models have been carried out by Kramer (1978a), Ball (1999), Kirchgassner (2000), Laussell and Le Breton (2002) and others.

The most important point made by the material discussed in this Section is that there are important conclusions for models of electoral competition with deterministic voting that can change when one introduces candidate uncertainty about voters' choices into the models.

9. Finite-dimensional models with probabilistic voting

9.a) Hinich's model

Hinich (1978) analyzed *both* unidimensional *and* multidimensional models with probabilistic voting. One of the things he did was identify conditions where, if a pure-strategy equilibrium exists, it must be at the mean. Sufficient conditions for the existence of such a pure-strategy equilibrium were also presented.

Hinich assumed that $X = \mathbb{R}^h$ (without restricting h to be 1). He also assumed that the set of voters is finite. In addition, he assumed that, for each $\in$, there is a most preferred alternative m( ) $\in$ X, and an (h x h), symmetric, positive-definite matrix, A( ), which enter into 's evaluation of each candidate's strategy. More specifically, he assumed that they enter into the determination of a policy-related "loss" (or negative utility) associated with the winning candidate's choice of a particular x $\in$ X. In particular, Hinich assumed that this loss is the number assigned by the following function (which depends on the "distance" between s and m( )):

$$L_\omega(s) = M(\|s - m( )\|_{A(\cdot)}) \tag{9.a.1}$$

where M is a monotonically increasing function and $\|y\|_A = [y^t A y]^{½}$. He also assumed that, for each $\in$, there is a nonpolicy loss, e_1( ), for if candidate 1 is elected and a nonpolicy loss, e_2( ), for if candidate 2 is the winner.

Hinich assumed that the candidates are uncertain about the choices that the voters are going to make on election day because the candidates are uncertain about the following characteristics for any given $\in$: the most-preferred alternative m( ); the matrix A( ); the non-policy losses, e_1( ) and e_2( ). This uncertainty was formulated by assuming that, for any given $\in$, the candidates' expectations are given by a random variable on a set of possible values for m( ), A( ), e_1( ) and e_2( ). So the formulation of a candidate's expectations about a voter's characteristics is like the standard formulation of expectations about a statistical observation prior to it being randomly selected from a population with a known distribution.

Hinich assumed that, for any given pair of strategy choices $(s_1, s_2) \in S_1 \times S_2$, the probability that candidate 1 will get the vote of a particular individual (conditional on the voter having a particular most-preferred alternative m, matrix A, and nonpolicy values e_1 and e_2) is

$$P^1_\omega(s_1, s_2 \mid m( ) = m, A( ) = A, e_1( ) = e_1, e_2( )=e_2)$$

$$= \begin{cases} 1 \text{ if } M(\|s_1 - m\|_A) + e_1 < M(\|s_2 - m\|_A) + e_2 \\ 0 \text{ otherwise} \end{cases} \tag{9.a.2}$$

That is, will vote for candidate 1 if and only if his total loss (his policy-related loss plus nonpolicy loss) from having candidate 1 elected is smaller than his total loss from having candidate 2 elected. An analogous assumption (with 2 replacing 1 and 1 replacing 2, on the right-hand side of (9.a.2)) was made about the conditional probability that will vote for candidate 2 at any particular strategy $(s_1, s_2) \in S_1 \times S_2$. Equation (9.a.2) can, of course, be rewritten as

$$Pr^1_\omega(s_1, s_2 \mid m( ) = m, A( ) = A, \ (i) = \ ) = \begin{cases} 1 \text{ if } M(\|s_2 - m\|_A) - M(\|s_1 - m\|_A) > \varepsilon \\ 0 \text{ otherwise} \end{cases} \tag{9.a.3}$$

where $= e_1 - e_2$. Hinich denoted the conditional distribution function for (given m and A) by F_ε. Using this notation, (9.a.3) leads to the conclusion that

$$Pr^1_\omega(s_1, s_2 \mid m(\ ) = m, A(\ ) = A) = F_\varepsilon[M(\|s_2 - m\|_A) - M(\|s_1 - m\|_A)] \qquad (9.a.4)$$

for each possible s_1, s_2, m and A. He assumed as well that F_ε has a continuous density function f_ε.

These assumptions imply that, at any given $(s_1, s_2) \in S_1 \times S_2$, the expected vote for candidate 1 is

$$EV^1(s_1, s_2) = (\# \quad \cdot (E\{F_\varepsilon[M(\|s_2 - m\|_A) - M(\|s_1 - m\|_A)]\}) \qquad (9.a.5)$$

with the expected value on the right specifically being taken with respect to the joint distribution of m and A. (9.a.5) leads to

$$P\ell^1(s_1, s_2) = (\# \quad \cdot (2 \cdot E\{F_\varepsilon[M(\|s_2 - m\|_A) - M(\|s_1 - m\|_A)]\} - 1) \qquad (9.a.6)$$

at each $(s_1, s_2) \in S_1 \times S_2$ (as in Hinich's equation (5) (1978: p. 364)).

In his analysis of the resulting game for the candidates, Hinich considered two models that satisfy his assumptions. The first is the "absolute value model," where $M(y) = |y|$. Hinich pointed out that in the absolute value model, when X has one dimension, each candidate choosing a median most-preferred alternative is a pure-strategy equilibrium.

The second model that Hinich considered is the "quadratic model," where $M(y) = y^2$. Hinich (1978: p. 365) established the following result for the quadratic model:

Theorem: Consider the quadratic model and assume that $f_\varepsilon > 0$ with positive probability for all $(s_1, s_2) \in S_1 \times S_2$. If a pure-strategy equilibrium exists, then both candidates choose

$$= [E\{f_\varepsilon(0)A\}]^{-1}E\{f_\varepsilon(0)Am\} \qquad (9.a.7)$$

Hinich used this result to identify conditions where, if a pure-strategy equilibrium exists, it must be at the mean. In particular, he pointed out that if (a) $f_\varepsilon(0)$ is independent of m and A and (b) m and A are uncorrelated, then is the mean ideal point.

Hinich also obtained a stronger result for quadratic models where "f_ε is a normal density whose mean is zero and whose variance σ^2 is small." In particular, building on the previous theorem, Hinich (1978: p. 368) established the following result for unidimensional election models.

Theorem: Let p(m) be a density function for the voters' most-preferred alternatives. Assume there is an interval [a, b] such that $p(m) = 0, \forall\, m \notin [a, b]$ and $p(m) > 0, \forall\, m \in [a, b]$. Consider the quadratic model where has a normal distribution with mean 0 and variance ^{2}. There exists > 0 such that if < then

(i) a pure-strategy equilibrium exists, and

(ii) (s_1*, s_2*) is a pure-strategy equilibrium if and only if s_1* = s_2* = .

Hinich pointed out that, "for a one dimensional space, we know that the median ideal point is a global equilibrium when σ = 0" (p. 367). He then observed that his theorem "shows that there is a discontinuity in the equilibrium of the expected plurality game as σ → 0; The equilibrium jumps from α to the median as σ hits zero" (p. 368). This led Hinich to conclude that "a small amount of error in the choice rule is sufficient to destroy the generality and elegance of the Black-Downs unidimensional deterministic result" (p. 370).

In this analysis, Hinich focused on cases where σ is small -- since he wanted to compare the equilibria in deterministic voting models with the equilibria in "nearby" probabilistic voting models (with σ serving as his measure of proximity). Hinich's analysis did not provide results for cases where σ isn't small.

At the end of the theoretical analysis in his paper, Hinich concluded: "Unless the reader is willing to accept either the quadratic or the absolute value model, it is difficult to say anything about the outcome of majority rule voting using the spatial model with the uncertainty element in it" (1978: p. 370).

9.b) Expectations based on a binary Luce model

Considering the implications of candidates having uncertainty about voters' choices naturally raises the more general question of how one should model individuals' choices when there is uncertainty about what the individuals will choose. Mathematical psychologists and others have provided useful ways of thinking about this question (see, for instance, Luce and Suppes (1965), Krantz et al (1971) or Roberts (1979)). As is well known, the most famous model for such situations is (using the terminology of Becker, DeGroot and Marschak (1963: p. 43)) the "Luce model," which was originally developed by Luce (1959). This model and variations on it have served as the basis for statistical models of paired comparisons (see, for instance, Bradley (1985)).

When each voter is assumed to vote (and, hence, is simply deciding whether to vote for candidate 1 or candidate 2), he is making a type of paired comparison. One of the elements (in the pair that is being compared) is: candidate 1 and the strategy which candidate 1 has chosen. The other element is: candidate 2 and the strategy which candidate 2 has chosen. In this particular setting, the appropriate version of Luce's model is (again using the terminology of Becker, DeGroot and Marschak (1963: p. 44)) the "binary Luce model". Stated in the context of electoral competition models: The binary Luce model for the individuals' selection probabilities assumes: For each $\in$, there exists a positive, real-valued "scaling function", $f_\omega(x)$, on X such that

$$Pr^1_{\omega}(s_1, s_2) = \frac{f_{\omega}(s_1)}{f_{\omega}(s_1) + f_{\omega}(s_2)} \tag{9.b.1}$$

and

$$Pr^2_{\omega}(s_1, s_2) = \frac{f_{\omega}(s_2)}{f_{\omega}(s_1) + f_{\omega}(s_2)} \tag{9.b.2}$$

for each $(s_1, s_2) \in S_1 \times S_2$.

A number of authors who have analyzed Luce models have suggested that the scaling function used for a particular individual could be taken to be a utility function for that individual (see, for instance, Luce and Suppes (1965: p. 335)). That is, (in the notation used in this chapter) $f_\omega \equiv U_\omega, \forall \quad \in$

Coughlin and Nitzan (1981) proved the following result for electoral competition models with probabilistic voting which satisfy these two assumptions (along with some other assumptions, which are stated in the premise of the theorem).

Theorem: Suppose $(S_1, S_2; P\ell^1, P\ell^2)$ satisfies the following assumptions: (i) is a finite set, (ii) X is a non-empty, compact, convex subset of $\mathbb{R}^h$ (where h is a positive integer), (iii) each $\quad \in \quad$ has a positive, concave, differentiable, ratio-scale utility function U_ω on X and (iv) for each $\quad \in \quad$,

$$Pr^1_{\omega}(s_1, s_2) = \frac{U_{\omega}(s_1)}{U_{\omega}(s_1) + U_{\omega}(s_2)} \tag{9.b.3}$$

and

$$Pr^2_{\omega}(s_1, s_2) = \frac{U_{\omega}(s_2)}{U_{\omega}(s_1) + U_{\omega}(s_2)} \tag{9.b.4}$$

for each $(s_1, s_2) \in S_1 \times S_2$. Then (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if s_1^* and s_2^* maximize

$$W(x) = \sum_{\omega=1}^{n} \ln(U_\omega(x)) \tag{9.b.5}$$

on X (where ln(v) denotes the natural logarithm of v).

What follows is an outline of the proof that is in Coughlin and Nitzan (1981).

1. Since (S_1, S_2; $P\ell^1$, $P\ell^2$) is zero-sum and symmetric, (s, t) is a pure-strategy equilibrium if and only if (s, s) and (t, t) are pure-strategy equilibria.
2. Since the game is symmetric, (,) is a pure-strategy equilibrium if and only if $P\ell^1(x, \ )$ achieves a global maximum at x = .
3. The premise for the theorem implies $P\ell^1(x, \ )$ is concave in x. Hence, since X is convex, $P\ell^1(x, \ )$ achieves a global maximum at x = if and only if $P\ell^1(x, \ )$ achieves a local maximum at x = .
4. The premise for the theorem implies W(x) is concave in x. Hence, since X is convex, W(x) achieves a global maximum at x = if and only if W(x) achieves a local maximum at x = .
5. Evaluating the gradients of $P\ell^1(x, \ )$ and W(x) at x = establishes that

$$\nabla_x P\ell^1(x, \ )]_{x = \psi} = (1/2) \cdot \nabla W(x)]_{x = \psi} \tag{9.b.6}$$

6. Using (9.b.6), it follows that $P\ell^1(x, \ )$ achieves a local maximum at x = if and only if W(x) achieves a local maximum at x = .
7. By the conclusions in 3, 4 and 6, $P\ell^1(x, \ )$ achieves a global maximum at x = if and only if W(x) achieves a global maximum at x = .
8. By the conclusions in 2 and 7, (,) is a pure-strategy equilibrium if and only if W(x) achieves a global maximum at x = .
9. By the conclusions in 8 and 1, (s, t) is a pure-strategy equilibrium if and only if W achieves a maximum at s and at t.

Coughlin and Nitzan (1981) pointed out that, under the premise for the theorem, the following result holds.

Corollary: There is a pure-strategy equilibrium.

Under the premise of the theorem, there can be more than one pure-strategy equilibrium. But Coughlin and Nitzan (1981) also pointed out the following implication of the theorem (which identifies sufficient conditions for uniqueness).

Corollary: If at least one voter has a strictly concave utility function, then there is a unique pure-strategy equilibrium.

The theorem also establishes a connection between pure-strategy equilibria in electoral competitions and a social welfare function which has been analyzed by Sen (1970: Chapter 8*), Kaneko and Nakamura (1979) and others. Suppose that (a) there is a distinguished alternative x_0 (which is not in X) that is one of the worst possible alternatives for all individuals and (b) we set $U_\omega(x_0) = 0, \forall \ \in \ $. When this assumption is added to the premise for the theorem, (9.b.5) is a "Nash social welfare function" (see Kaneko and Nakamura (1979: p. 432)). In addition, the theorem implies that (s_1^*, s_2^*) is a pure-strategy equilibrium if and only if s_1^* and s_2^* maximize this Nash social welfare function.

It will be useful to contrast the implications of these results with the ones obtained when there is deterministic voting under the following assumptions: (i) is a finite set, (ii) X is a non-empty, compact, convex subset of $\mathbb{R}^1$ and (iii) each $\in$ has a positive, strictly concave, differentiable, ratio-scale utility function U_ω on X. Under these assumptions, for each $\omega \in \Omega$, the preference ordering $\succeq_\omega$ is single-peaked (with respect to the ordering relation "is less than or equal to" for real numbers). So the last theorem in Section 5 (that is the theorem in Section 5 for electoral competitions with a finite set of voters) implies: x is an equilibrium strategy for a candidate if and only if x is a median for the distribution of most-preferred alternatives (with respect to $\leq$).

First consider the cases where $\#\Omega$ is <u>odd</u>. When there is deterministic voting:
a) there is exactly one equilibrium in pure strategies, and
b) at the equilibrium in pure strategies, each candidate chooses the unique median for the distribution of most-preferred alternatives.
In the probabilistic voting model with (9.b.3) and (9.b.3) satisfied: a) there is exactly one equilibrium in pure strategies, and b) at the equilibrium in pure strategies, each candidate chooses the unique alternative that maximizes the Nash social welfare function. So, both models have a unique prediction for the candidates' equilibrium strategies, *but* the predictions don't match unless the alternative that maximizes the Nash social welfare function happens to coincide with the median for the distribution of most-preferred alternatives. So, when $\#\Omega$ is <u>odd</u>, *in most cases* the equilibrium strategy in the probabilistic voting model with (9.b.3) and (9.b.3) satisfied will **not** be a median.

Now consider the cases where $\#\Omega$ is <u>even</u>. When there is deterministic voting:
1) If there is a unique median for the distribution of most-preferred alternatives, then
a) there is exactly one equilibrium in pure strategies, and
b) at the equilibrium in pure strategies, each candidate chooses the unique median for the distribution of most-preferred alternatives.
2) If there isn't a unique median for the distribution of most-preferred alternatives, then
a) there is more than one equilibrium in pure strategies, and
b) at each equilibrium in pure strategies, each candidate chooses a median for the distribution of most-preferred alternatives.
Once again, in the probabilistic voting model with (9.b.3) and (9.b.3) satisfied: a) there is exactly one equilibrium in pure strategies, and b) at the equilibrium in pure strategies, each candidate chooses the unique alternative that maximizes the Nash social welfare function. The comparison with 1) is the same as when $\#\Omega$ is <u>odd</u>. For 2), there are two noteworthy comparisons. First: Only the probabilistic voting model with (9.b.3) and (9.b.3) satisfied has a unique prediction for the candidates' equilibrium strategies. Second: Like when $\#\Omega$ is <u>odd</u>, *in most cases* the equilibrium strategy in the probabilistic voting model with (9.b.3) and (9.b.3) satisfied will **not** be a median.

Samuelson (1984) subsequently studied election models in which the candidates use a binary Luce model, with the added feature that the candidates' strategies are restricted. The restrictions on the candidates' strategies were specifically included by assuming that each candidate has (a) an initial position, $w_c \in X$ and (b) a nonempty, compact, convex set, $S_c(w_c) \subseteq X$, of feasible options open to him. Samuelson also assumed (a) there is a nonempty, compact, Euclidean set of possible voter characteristics, (b) each scaling function is a concave function of the possible candidate strategies, (c) each scaling function is a continuous function of *both* the possible candidate strategies *and* the possible voter characteristics and (d) the electorate can be summarized by a continuous density function on the set of possible voter characteristics. Samuelson (1984: p. 311) established that the resulting model has the following property: For each $(S_1(w_1), S_2(w_2); P\ell^1, P\ell^2)$ there exists a pure-strategy equilibrium.

Samuelson used his result to analyze (a) a sequence of elections in which a series of opposition candidates challenged incumbents and (b) the apparent incumbency advantage that has been observed in recent congressional elections.

In light of the discussion in Subsection 9.a), a natural question is whether there are any noteworthy connections between the models discussed in this Subsection and models in which voters have additively separable loss functions like the ones studied by Hinich (1978). It is known that results about binary Luce models have direct implications for models in which utility/loss can be written as the sum of a nonrandom utility/loss function and a random "error" term (see, for instance, Luce and Suppes (1965: Section 5.2)). The established connection between these alternative models can be used in the context of electoral competition models as follows. Suppose that (a) each voter, , has a policy-related loss function, $L_\omega(s) = -\log(f_\omega(s)) - b_\omega$ (where f_ω is a positive, real-valued function on X and b is a constant), (b) analogous to (9.a.3), for each and each $(s_1, s_2) \in S_1 \times S_2$,

$$\mathbf{Pr}^1_\omega(s_1, s_2) = \begin{cases} \mathbf{1 \text{ if } } L_\omega(s_2) - L_\omega(s_1) > \varepsilon_\omega \\ \mathbf{0 \text{ otherwise}} \end{cases} \tag{9.b.7}$$

and (c) $_\omega$ has a logistic distribution. Then, using the argument in the proof of Luce and Suppes' Theorem 30 (1965: p. 335) it follows that the candidates are using a binary Luce model. Therefore, when the remaining assumptions for the theorem in this Subsection are also made, the conclusion of that theorem holds for the corresponding model with separable policy-related and non-policy voter utilities/losses.

9.c) Lindbeck and Weibull's model

Lindbeck and Weibull developed a model of "balanced-budget redistribution between socio-economic groups as the outcome of electoral competition between two political parties" (1987: p. 273) in which the parties have "incomplete information as to political preferences . . . related to ideological considerations and politicians personalities" (p. 274). Since they modeled redistribution between groups, in each case where there are three or more groups, the strategy set for the parties is multidimensional. As a consequence, as with the multidimensional election models discussed in Subsection 9.b), "the presence of uncertainty is crucial for *existence* of equilibrium in [their] model" (p. 280).

Lindbeck and Weibull assumed that the set of voters, , is finite and that each $\in$ has a fixed gross income, $Y_\omega \in \mathbb{R}^1_{++}$. They also assumed that the candidates have a (common) partition, $= \{1, \ldots, m\}$, of the electorate (with $m \geq 2$). Note that in what follows, the elements in will be used as indices for the groups as well as to denote the sets of voters that constitute the candidates' partition. For each $\in$, n_θ will denote the number of voters in .

Lindbeck and Weibull assumed that the strategies available to the candidates are vectors, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, of possible transfers to the members of the m groups. In addition, they assumed that each candidate must select a balanced-budget redistribution in which each individual's net income must be positive. Hence

$$X = \{x \in \mathbb{R}^m : \sum_{\theta=1}^{m} n_\theta \cdot x_\theta = 0 \;\; \& \;\; Y_\omega + x_\theta > 0, \forall \quad \in \quad , \forall \quad \in \quad \}. \tag{9.c.1}$$

As in many of the analyses that have already been discussed, Lindbeck and Weibull assumed that any given voter's utility for a particular candidate's election is the *sum of* his utility for the candidate's strategy *and* an additional component that reflects "other factors" that affect his preferences for the candidates. For each $\in$, his "final" (or "net") income will be $c_\omega = Y_\omega + x_\theta$ where is the group which contains . They explicitly assumed that each voter has a twice-differentiable utility function, $v_\omega(c_\omega)$. Using this notation, any particular 's utility function on X can be written as

$$U_\omega(x) = v_\omega(Y_\omega + x_\theta). \tag{9.c.2}$$

where is the group which contains . Lindbeck and Weibull assumed that, for each $\in$,

$$v_\omega^{\prime}(z) > 0 \;\; \& \;\; v_\omega^{\prime\prime}(z) < 0, \forall\, z > 0 \tag{9.c.3}$$

and

$$\lim_{z \to 0} v_\omega^{\prime}(z) = +\infty \;\; \& \;\; \lim_{z \to \infty} v_\omega^{\prime\prime}(z) = 0. \tag{9.c.4}$$

Lindbeck and Weibull assumed that, for each $\in$ and each $(s_1, s_2) \in S_1 \times S_2$,

$$\mathbf{Pr}^1{}_{\omega}(\mathbf{s}_1, \mathbf{s}_2) = \begin{cases} \mathbf{1} \ \textbf{if} \ \mathbf{U}_{\omega}(\mathbf{s}_1) - \mathbf{U}_{\omega}(\mathbf{s}_2) > \mathbf{a}_{\omega} - \mathbf{b}_{\omega} \\ \mathbf{0} \ \textbf{otherwise} \end{cases} \tag{9.c.5}$$

where "a_ω is the utility that individual derives from other policies in candidate [1's] political program and likewise with b_ω [and candidate 2]" (p. 276). They also made an analogous assumption about $Pr^2{}_\omega(s_1, s_2)$ (with *both* 1 and 2 *and* a and b interchanged on the right-hand side of (9.c.5) or, equivalently, with the inequality on the right-hand side of (9.c.5) reversed).

Lindbeck and Weibull assumed that the two parties treat a_ω and b_ω as random variables. More specifically, they assumed that the parties have a twice continuously differentiable probability distribution, F_ω, for $b_\omega - a_\omega$. Letting $f_\omega(y) = F'_\omega(y)$, Lindbeck and Weibull additionally assumed that $f_\omega(y) > 0, \forall\, y \in \mathbb{R}^1$.

One of the things that Lindbeck and Weibull did was identify a necessary condition for a pure-strategy equilibrium in their model (p. 278). Their necessary condition is stated in the following theorem.

Theorem: If (s_1, s_2) is a pure-strategy equilibrium, then

(i) $s_1 = s_2 \equiv s^*$, and

(ii) there exists $\lambda > 0$ such that, for each $\in$

$$\sum_{w \in \theta} v_\omega'(Y_\omega + s^*) \cdot f_\omega(0) = n_\theta \cdot \lambda \tag{9.c.6}$$

Equation (9.c.6) is of particular significance because it is also a first-order necessary condition for maximizing the weighted Benthamite social welfare function

$$W(x) = \sum_{\omega=1}^{n} v_\omega(c_\omega) \cdot f_\omega(0) \tag{9.c.7}$$

on the set X. Lindbeck and Weibull pointed out that, if the parties use the same candidate preference distribution for each voter (that is, $F_\alpha = F_\beta, \forall \ , \ \in \ $), then "in this special case democratic electoral competition for the votes of selfish individuals produces the same income distribution as would an omnipotent Benthamite government" (p. 278).

In addition, Lindbeck and Weibull (1987: p. 280) identified a sufficient condition for the existence of a unique pure-strategy equilibrium in their model. Their condition used a "concavity index" from Debreu and Koopmans (1982: p.27): $\iota(v_\omega) = \inf_r\{|v_\omega''(r)|/(v_\omega'(r))^2\}$.

Theorem: If

$$\sup_{t} \{|f_{\omega}^{\prime}(t)|/f_{\omega}(t)\} \leq \iota(v_{\omega}), \forall \quad \in \tag{9.c.8}$$

then a unique pure-strategy equilibrium exists.

They also drew attention to the fact that (9.c.8) is “more easily satisfied the larger is the degree of uncertainty” (p. 281).

Lindbeck and Weibull also examined what happens to the results stated above when additional (or alternative) features are included in their model. The particular extensions that they considered were (a) administration costs (which could vary from group to group), (b) abstentions, (c) a role for party activists who do more for a candidate than simply vote for him, and (d) each candidate’s wanting to maximize his probability of winning (rather than his expected plurality). The conclusions that they arrived at were very similar (albeit not identical) to the two theorems stated above. The minor differences that result from using these alternative assumptions were discussed in detail in the corresponding sections in Lindbeck and Weibull’s article.

10. Probabilistic voting models with fixed candidate characteristics and voter predictions

10.a) Fixed candidate characteristics

Enelow and Hinich (1982) analyzed an election model in which fixed candidate characteristics (that is, characteristics which the candidates can’t alter during the election being considered) are important. The particular model that they analyzed is very similar to the one in Hinich (1978) (which was discussed in Subsection 9.a). In particular, as the ensuing discussion will make clear, most of their assumptions correspond to ones that were made by Hinich en route to his existence theorem. However, their existence result (the theorem in this section) is *not* a special case for Hinich’s earlier existence theorem. Thus their analysis succeeded in identifying a new sufficient condition for the existence of an electoral equilibrium.

Enelow and Hinich (1982) assumed that the (common) strategy set, X, for the candidates is such that $[0, 1] \subset X \subseteq \mathbb{R}^1$ -- with the positions in X interpreted as expenditure levels on a single public-spending issue. As did Hinich (1978), they assumed that, for each , there is an most-preferred alternative, m( ) $\in$ X, that enters into ’s evaluation of each candidate’s strategy. One part of the premise for Hinich’s existence theorem is that there is a density function, p(m), for the voter most-preferred alternatives such that (a) p(m) = 0 for m outside an interval [a, b] and (b) p(m) > 0 inside. Enelow and Hinich (1982), by contrast, assumed that the electorate can be partitioned into two groups, = 1 and = 2 (with n_1 and n_2 voters, respectively), and that (a) each $\in$ = 1 has the most-preferred alternative m( ) = 0 and (b) each $\in$ = 2 has the most-preferred alternative m( ) = 1.

Enelow and Hinich (1982) assumed that each makes a numerical assessment of the nonspatial characteristics of candidate 1 (and also the nonspatial characteristics of candidate 2), denoted by $\chi_{\omega 1}$ (respectively, $_{\omega 2}$). This assessment was supplemented by a positive, numerical measure of the importance that attaches to the candidates' strategies (relative to their nonpolicy characteristics), denoted by a_ω. They assumed that, for each and each $(s_1, s_2) \in S_1 \times S_2$,

$$\Pr^1_\omega(s_1,s_2) = \begin{cases} 1 \text{ if } ((s_2 - m(\omega)))^2 - ((s_1 - m(\omega)))^2 > \varepsilon_\omega \\ 0 \text{ otherwise} \end{cases} \tag{10.a.1}$$

where $_\omega = (\ _{\omega 2}/a_\omega) - (\ _{\omega 1}/a_\omega)$. An analogous assumption (with 2 replacing 1, and 1 replacing 2, on the right-hand side of (10.a.1) and in the definition of $_\omega$) was made about $\Pr^2_\omega(s_1, s_2)$. Note that in (10.a.1) (a) $(s_c - m(\ ))^2$ corresponds to Hinich's loss function (being, in particular, the special case of Hinich's quadratic model that arises when $X \subseteq \mathbb{R}^1$ and $A = [1]$) and (b) $-\ _{\omega 1}/a_\omega$ and $-\ _{\omega 2}/a_\omega$ correspond to e_1 and e_2, respectively, in Hinich (1978). Or, equivalently (translating this observation from "loss" terms into "utility" terms), $U_\omega(x) = -(x - m(\ ))^2$ can be thought of as a policy-related utility function for voter and $(\ _{\omega 1}/a_\cdot)$ (or $(\ _{\omega 2}/a_\cdot)$) can be thought of as a nonpolicy value for if candidate 1 (respectively, candidate 2) wins the election.

In his existence theorem, Hinich (1978) assumed that the variance, ^{2}, for the nonpolicy difference, $_\omega$, is the same for each voter. Enelow and Hinich (1982), on the other hand, assumed that the candidates believe that (a) the distribution of $_\omega$ *across the group* = 1 is normal with mean 0 and variance $_1^2$, and (b) the distribution of $_\omega$ *across the group* = 2 is normal with mean 0 and variance $_2^2$ (as the notation suggests, $_2$ need not equal $_1$). This distributional assumption is consistent with assuming that the candidates know the nonpolicy difference for each voter and is, alternatively, also consistent with assuming that they are uncertain about the nonpolicy value for any particular voter but (nonetheless) know the distribution of the nonpolicy values across each group.

In order to be able to state their result in a relatively simple way, Enelow and Hinich (1982) used the following notation:

$$\bar{x} \equiv [n_2 \cdot\ _1] / [n_1 \cdot\ _2 + n_2 \cdot\ _1]. \tag{10.a.2}$$

The number, $\bar{x}$, for any special case is, as they pointed out, a weighted mean most-preferred alternative (which has the property that, *if* $_1 =\ _2$, *then* $\bar{x}$ is the unweighted mean most-preferred alternative). Using this notation, the existence result in Enelow and Hinich (1982: pp. 123-4) can be stated as follows.

Theorem: If $_1^2 > 2 \cdot (1 - \bar{x})^2$ and $_2^2 > 2 \cdot [1 - (1 - \bar{x})^2]$,

(i) an equilibrium exists, and

(ii) (s_1^*, s_2^*) is an equilibrium if and only if $s_1^* = s_2^* = \bar{x}$.

This result is similar to Hinich's (1978) existence result in that the conclusion is: A unique equilibrium exists and is located at a weighted mean of the voter most-preferred alternatives. In addition to the differences in the underlying assumptions that were noted above, Enelow and Hinich's (1982) result also differs from Hinich's (1978) existence theorem in that (a) the sufficient condition involves inequalities that require relatively larger variances for the nonpolicy values (rather than a relatively small variance) and (b) the result provides precise inequalities for the variances (rather than just assuring that an appropriate inequality exists).

Enelow and Hinich (p. 127) argued that σ_1^2 and σ_2^2 "will be of sufficient size in a large society" for their premise to be satisfied. In particular, they suggested that this will be the case "since the size of these parameters is a function of the number of different views concerning relative differences between candidates' nonspatial characteristics and issue salience." They did not identify what happens if the variances aren't large enough to satisfy the inequalities in their premise.

Enelow and Hinich argued that it is especially significant that the equilibrium in their election model is at a "compromise" position, rather than at the most-preferred alternative of one of the two groups. They pointed out that this property implies that (in their model) the minority group has some influence over the outcome and, therefore, there is not a "tyranny of the majority." They argued that this implication reflects an important difference between (a) representative democracy (where the citizens vote for candidates who both propose policies and have fixed characteristics that the voters care about) and (b) direct democracy (where the citizens vote solely on policies).

10.b) Predictive maps

Enelow and Hinich (1984: Sections 5.1, 5.2 and 5.4) extended their earlier analysis of election models with fixed candidate characteristics by considering cases where there are "predictive mappings" for the voters. These mappings, which are discussed in detail in Enelow and Hinich (1984: Chapter 4), allow each voter to map *from* a candidate's strategy *to* the policies he thinks will actually be adopted if the candidate is elected. The predictive mappings, therefore, allow for the possibility that voters will interpret candidates' strategies in their own individual ways, instead of simply believing that each candidate will carry out the policies that he advocates during the election.

Enelow and Hinich assumed that the candidates' strategy set is the closed interval [-1/2, +1/2] (1984: pp. 101-2 in Appendix 5.1). As before, they assumed that there are two groups, $= 1$ and $= 2$ (with n_1 and n_2 voters, respectively). They made two particular assumptions about the groups: (a) they partition the electorate and (b) for each $\in \ = \{1,2\}$, the voters in the group have a common most-preferred alternative, $m_\theta \in \mathbb{R}^2$, and a common predictive map,

$$w(s_c) = b_\theta + s_c \cdot v_\theta \tag{10.b.1}$$

(with domain [-1/2, +1/2]). Enelow and Hinich specifically assumed that each predictive map goes from the unidimensional strategy adopted by a candidate to a two-component vector of "predicted policies." Thus, for any given $(s_1, s_2) \in S_1 \times S_2$, the corresponding $w_\omega(s_1) \in \mathbb{R}^2$ (and the corresponding $w_\omega(s_2) \in \mathbb{R}^2$) is the two-dimensional vector of policies that each person in   thinks will actually be adopted if candidate 1 (respectively candidate 2) is elected. Note that, since the candidates' strategies are in $\mathbb{R}^1$ and the predicted policies are in $\mathbb{R}^2$, the b_θ and v_θ parameters in (10.b.1) are two-component vectors, which can therefore be written as $b_\theta = (b_{\theta 1}, b_{\theta 2})^t$ and $v_\theta = (v_{\theta 1}, v_{\theta 2})^t$. Similarly, the common most-preferred alternative for the individuals in a given group, , can be written as $m_\theta = (m_{\theta 1}, m_{\theta 2})^t$.

This time around, Enelow and Hinich (1984: p. 83) explicitly interpreted the groups in their model as "interest groups," making the following argument for this interpretation:

> "The . . . homogeneous groups can be thought of as the politically salient interest groups in the electorate. Each interest group has a common set of policy concerns and looks at the candidates the same way. . . . This conception of an interest group is particularly appropriate from the point of view of the candidates. It is a common practice in campaigns to view the electorate as being composed of homogeneous issue groups. . . . This practice is a shorthand device that permits candidates to plan campaign strategies without becoming lost in the complexities of individual voter attitudes."

As before, they also assumed that each voter places a nonpolicy value on candidate 1's winning the election and a nonpolicy value on candidate 2's winning. Accordingly, analogous to equation (10.a..1), Enelow and Hinich assumed that, for each   and each $(s_1, s_2) \in S_1 \times S_2$,

$$Pr^1_\omega(s_1, s_2) = \begin{cases} 1 & \text{if } \|w_\omega(s_2) - m_\theta\|_2^2 - \|w_\omega(s_1) - m_\theta\|_2^2 > \varepsilon_\omega \\ 0 & \text{otherwise} \end{cases} \tag{10.b.2}$$

where (a) $\|y\|_2 = (y^t I_2 y)^{1/2}$ and (b) $_\omega$ is (again) the *difference between* the nonpolicy value for   if candidate 2 is elected *and* the nonpolicy value for   if candidate 1 is elected. An analogous assumptions (with the inequality on the right-hand side of (10.b.2) reversed) was made about $Pr^2_\omega(s_1, s_2)$ (with 1 and 2 interchanged). In addition, Enelow and Hinich assumed that the candidates believe that the distribution of $_\omega$ *across a given interest group*   is normal with mean zero and variance $_\theta^2$.

For each group , Enelow and Hinich (p. 102) used B_θ to denote any number which is greater than or equal to both $(v^1_{\theta 1} + v^2_{\theta 2})^{1/2}$ and $[(m_{\theta 1} - b_{\theta 1})^2 + (m_{\theta 2} - b_{\theta 2})^2]^{1/2}$ (the elements that appear in these two equations are specifically the corresponding components in the vectors v_θ, m_θ, b_θ). For instance, for each , B_θ could simply be the larger of the two numbers. Enelow and Hinich (1984: pp. 85-6, 101-2) established:

Theorem: If ${}_1 > 3 \cdot (1.5)^{1/2} \cdot (B_1)^2$ and ${}_2 > 3 \cdot (1.5)^{1/2} \cdot (B_2)^2$, then (i) an electoral equilibrium exists, and (ii) (s_1^*, s_2^*) is an electoral equilibrium if and only if $s_1^* = s_2^*$

$$= \frac{\left\{ n_1 \sigma_1^{-1} \left[v_{11}(m_{11} - b_{11}) + v_{12}(m_{12} - b_{12}) \right] + n_2 \sigma_2^{-1} \left[v_{21}(m_{21} - b_{21}) + v_{22}(m_{22} - b_{22}) \right] \right\}}{\left[n_1 \sigma_1^{-1} (v^2{}_{11} + v^2{}_{12}) + n_2 \sigma_2^{-1} (v^2{}_{21} + v^2{}_{22}) \right]} \qquad (10.b.3)$$

Enelow and Hinich (1984: pp. 87-8) supplemented their analysis with an example of an election model with fixed candidate characteristics, predictive maps and interest groups.

Example: Assume that group = 1 is twice as large as group = 2 (that is $n_1 = 2 \cdot n_2$). Also assume that each voter in group = 1 has the most-preferred alternative $m_1 = (.2, .8)^t$ and the predictive map

$$w_1(s) = \begin{bmatrix} +.3 \\ +.7 \end{bmatrix} + s \cdot \begin{bmatrix} +.2 \\ -.2 \end{bmatrix} \qquad (10.b.4)$$

and that each voter in group = 2 has the most-preferred alternative $m_2 = (.35, .65)^t$ and the predictive map

$$w_1(s) = \begin{bmatrix} +.2 \\ +.8 \end{bmatrix} + s \cdot \begin{bmatrix} +.3 \\ -.3 \end{bmatrix} \qquad (10.b.5)$$

Enelow and Hinich (1984) pointed out that their result directly implies that (in their example) if ${}_1 > 3 \cdot (1.5)^{1/2} \cdot B_1^2 = .294$ and ${}_2 > 3 \cdot (1.5)^{1/2} \cdot B_2^2 = .661$, *then* there is a unique equilibrium, $(s_1^*, s_2^*) \in S_1 \times S_2$, with

$$s_1^* = s_2^* = \frac{\left\{ -.08\sigma_1^{-1} + .09\sigma_2^{-1} \right\}}{\left[.16\sigma_1^{-1} + .18\sigma_2^{-1} \right]} \ . \qquad (10.b.6)$$

They also pointed out that *if* one makes the further assumption that ${}_1 = {}_2$, *then* their result provides the even more precise conclusion that $s_1^* = s_2^* = .03$.

Enelow and Hinich (1984: Section 5.4) also used this example to illustrate the fact that when their sufficient condition is *not* satisfied, their model with fixed candidate characteristics, predictive maps, and interest groups *need not* have an equilibrium. They did so by (more specifically) showing that, when this example is supplemented with the alternative assumption that ${}_2 = 0$ and $.06 < {}_1 < .12$, there is *no* electoral equilibrium.

10.c) Summing up

One reason why results that have been derived for probabilistic voting models are important is: Equilibrium existence results have been obtained for distributions of voter preferences that have no Nash equilibrium when voting is deterministic. This is illustrated by the theorems stated in Subsections 9.b) and 9.c). More specifically, the theorems apply to multidimensional strategy spaces, but do not require any symmetry properties for the distribution of voter preferences.

A second reason why results that have been derived for probabilistic voting models are important is: When there is a Nash equilibrium in a deterministic voting model, there may be a different Nash equilibrium for a corresponding probabilistic voting model. For example, consider the assumptions in the premise of the theorem stated in Subection 9.b). A relevant special case (for the assertion made at the beginning of this paragraph) is the one where $S \subseteq \mathbb{R}^1$. In this special case: A Nash equilibrium exists, with each candidate's strategy being a number which maximizes the Nash social welfare function in (9.b.5). If we now replace the assumption of probabilistic voting with the assumption of deterministic voting, then the theorem in Section 5 applies. Therefore a Nash equilibrium exists, with each candidate's strategy being a median for the distribution of most-preferred alternatives. However, the other assumptions that have been made do *not* imply that the median will maximize the Nash social welfare function in (9.b.5). There are some circumstance where there are equilibrium strategies for a deterministic voting model which are the same as the equilibrium strategies for a corresponding probabilistic voting model. But that is the exception, rather than the rule.

11. Alternative solution concepts

The *absence* of a pure-strategy equilibrium for a pair of political candidates could be described as follows: The process of trying to second-guess the other candidate is hopeless. For each possible alternative, the other candidate should select a different alternative. But, given this alternative, the first candidate should select yet another alternative. This description suggests that, when there is no pure-strategy equilibrium, there is no definite prediction about what will happen. This conclusion has led Ordeshook (1986) to argue that "it is unsatisfactory to conclude that pure-strategy equilibria need not exist in two-candidate elections, since this conclusion leaves us without any hypotheses about eventual strategies and outcomes for a wide class of situations" (p. 180).

Because pure-strategy equilibria are rare in multidimensional election models with deterministic voting, some authors have concluded that only a fundamental reformulation of how the competitive process works will allow for meaningful predictive deductions about candidate decisions in multidimensional election models. Some authors have proposed a solution concept which generalizes the notion of a pure-strategy equilibrium (by including pure-strategy equilibria in the set of solutions when they exist and allowing for a non-empty set of solutions when there is no pure-strategy equilibrium). Others give up on the idea of studying pure-strategy equilibria and use a different type of solution concept.

11.a) A dynamical approach

Some authors have concluded that analyses of electoral competition in the substantively important multidimensional case must be based on explicit hypotheses about disequilibrium behavior. For instance, Kramer (1977b) used hypotheses about disequilibrium choices by political candidates in an analysis of sequences of elections.

Kramer assumed that there is a finite set of voters. He assumed that there is a set of alternatives and that this set is a multidimensional Euclidean space. He also assumed that each voter has a most-preferred alternative and Euclidean preferences.

Kramer considered two political candidates that compete in a sequence of elections. In any given election, 1) each candidate competes by advocating a particular alternative, 2) there is deterministic voting and 3) if one of the candidates gets more votes than the other, it wins; otherwise the winner is selected by the toss of a fair coin. The winning candidate then enacts the alternative it advocated. In the next election, the incumbent must defend the alternative and the challenger chooses an alternative which maximizes its vote. A sequence of successively enacted alternatives which is generated by this process is called a "vote-maximizing sequence (or trajectory)". (x_t) will be used to denote any such sequence.

Kramer also used the following concepts, which had previously been developed by Simpson (1969). For any given $(X, (\succeq_1, ..., \succeq_{\#\Omega}))$, he considered the following function on X.

$$v(x) = \max_{y \in X} \#\{ \qquad : y \succ_\omega x\} \tag{11.a.1}$$

The number

$$v^* = \min_{x \in X} v(x) \tag{11.a.2}$$

is called the minmax number. The set

$$M(v^*) = \{x \in X : v(x) = v^*\} \tag{11.a.3}$$

is called the minmax set.

Kramer used the term "(x_t) enters $M(v^*)$" to mean $\exists$ t such that $x_t \in M(v^*)$. Saying (x_t) enters $M(v^*)$ in period t" will mean $x_t \in M(v^*)$ and $x_{t-1} \notin M(v^*)$. Kramer pointed out that, under the assumptions stated above, there exist vote-maximizing sequences of alternatives that do <u>not</u> enter the minmax set.

When a (weak) simple majority Condorcet winner exists, the minmax set is the set of (weak) simple majority Condorcet winners. So, when there is a pure-strategy equilibrium in the one-period game $(X,X;VS^1,VS^2)$, the minmax set is the set of equilibrium strategies. Hence in any such situation, if (x_t) enters $M(v^*)$ in period t then $x_r \in M(v^*)$ for each x_r in the sequence with $r > t$.

Kramer argued that, when a (weak) simple majority Condorcet winner doesn't exist, the elements of the minmax set are the ones which "most resemble" (weak) simple majority Condorcet winners. He also pointed out that, under the assumptions he considered, the minmax set is typically a small proper subset of the society's set of alternatives.

Kramer also used the following definition of the distance from a point $x \in \mathbb{R}^h$ to a set $Z \subset \mathbb{R}^h$:

$$d(x, Z) = \inf_{z \in Z} \|x - z\|_h \tag{11.a.4}$$

Kramer described the following theorem as "the main result" (p. 320) in his analysis.

Theorem: Suppose $(X, (\succeq_1,...,\succeq_{B'E}))$ is a regular model where $X = \mathbb{R}^h$ (with $\quad > h \geq 2$) and each voter has Euclidean preferences. Consider any vote-maximizing trajectory, (x_t). For any t where $x_t \notin M(v^*)$,

$$d(x_{t+1}, M(v^*)) < d(x_t, M(v^*)) \tag{11.a.5}$$

Kramer pointed out that his theorem implies that, for any vote-maximizing trajectory, the distance to the minmax set must be monotonically decreasing *until* the trajectory reaches the set (p. 320). He described this property of his model by stating that "every vote-maximizing trajectory will tend to approach the minmax set". Kramer also described this property by saying that every vote-maximizing trajectory "necessarily converges on the minmax set" (p. 323). However, the convergence is limited because 1) it is possible for a trajectory to never enter the minmax set, and 2) when there is no pure strategy equilibrium in $(X,X;VS^1,VS^2)$ there exist sequences of alternatives that enter the minmax set, but have infinite subsequences which are outside the minmax set (e.g., because, whenever the minmax set is reached, the next vote-maximizing proposal selected is outside the minmax set).

11.b) Mixed-strategy equilibria

The most prominent research on electoral competition using a solution concept which generalizes the notion of a pure-strategy equilibrium is research that has replaced the concept of a pure-strategy equilibrium for the candidates with the concept of a mixed-strategy equilibrium. The pioneering work on this topic was done by Shubik (1968; 1970; 1984: pp. 625-629).

Shubik derived the mixed-strategy equilibria for some specific electoral competitions. Additionally, he considered the relation between mixed-strategy equilibria and pairwise comparisns made with simple majority rule. His analysis of this relation, included the following (important) comment: "If there is a set of outcomes which are intransitive among themselves but are preferred to all others the resulting game of strict opposition between the two candidates will call for a mixed strategy over the set of outcomes which display the intransitivity" (p. 348).

The consequences of assuming that candidates may play mixed-strategies have also been explored by Kramer (1978b), Calvert (1986), Laffond, Laslier, and Le Breton (1993, 1994), Laslier (1997: Section 10.1; 2000), Dutta and Laslier (1999: Sections 3.3 and 4.3), Banks, Duggan and Le Breton (2002) and others. Some of the authors who have studied mixed-strategy equilibria have suggested that they could be interpreted as situations where, when the voters compare the candidates, the voters are comparing lotteries which correspond to the candidates' mixed strategies (e.g., Shubik (1968, 1970, 1984), Laffond, Laslier, and Le Breton (1994), and Laslier (2000)).

Critiques of the idea of using a mixed-strategy equilibrium as the equilibrium concept for election models have been been written by Riker and Ordeshook (1973: p. 340), McKelvey and Ordeshook (1976: pp. 1174-1175), Kramer (1977: pp. 695-696), Ordeshook (1986: pp. 180-182; 1992: pp. 107-108), Artale and Gruner (2000: pp. 20-29), Ansolabehere and Snyder (2000: p. 334) and others.

One line of criticism of mixed-strategy equilibria is based on the empirical observation that political candidates don't use random devices to select strategies. For instance, Ordeshook (1986: p. 181) states: "it seems silly to conceptualize candidates spinning spinners or rolling dice to choose policy platforms."

A second line of criticism is based on the idea that, even if the political candidates did use random devices to select pure strategies, they would have to communicate the pure strategies to the voters before election day. In this context, Ordeshook (1986) has suggested that it is useful to distinguish between a candidate using a "mixed strategy" and a candidate using a "risky strategy". His rationale is: "In an election context, if a candidate abides by some mixed strategy, then, ultimately, the electorate sees only the pure strategy he chooses by lottery. If a candidate adopts a risky strategy, though, then the electorate sees the lottery as that candidate's platform. Risky strategies are pure strategies" (p. 186). The second line of criticism argues that in cases where there is no pure-strategy equilibrium: 1) Once the "selected pure strategies" (that is, the ones which result from the candidates adopting mixed strategies and then using random devices to select pure strategies) are communicated to the voters, at least one of the candidates will be better of changing its pure strategy; 2) At the strategy pair that results from this change, at least one of the candidates will be better off changing its pure strategy; 3) The conclusion in 2) applies over and over again (ad infinitum). As a result there is no definite prediction about what will happen.

Artale and Gruner (2000) presented another argument against viewing a mixed-strategy equilibrium as a reasonable solution for an electoral competition. They first suggested (pp. 20-21) that a reasonable solution will reflect the following “stylized facts of political life in representative democracies”: the presence of “political stability” and “the absence of discrimination against single groups”. Then they considered the hypothesis that a mixed-strategy equilibrium is a reasonable solution for an electoral competition.

They began by deriving the mixed-strategy equilibrium for an electoral competition (with deterministic voting) where the alternatives are possible income distributions. By analyzing the mixed-strategy equilibrium, they were able to show that it (1) “cannot explain high degrees of political stability (p. 29) and (2) implies that one should “observe frequent cases of discrimination against single groups” (p. 29). Since these properties don’t match with their stylized facts, they rejected the hypothesis that a mixed-strategy equilibrium is a reasonable solution for an electoral competition.

The preceding arguments against viewing a mixed-strategy equilibrium as a reasonable solution for an electoral competition interpret a mixed-strategy as a “chance mechanism” for selecting a pure strategy. Calvert (1986: Section 4.6) has suggested that Harsanyi’s (1973) alternative interpretation of a mixed-strategy equilibrium may provide a more useful way to think about mixed-strategies in an electoral competition.

In Harsanyi’s view, mixed-strategies are not the results of deliberate randomization. Rather, in his view, they result from imperfect knowledge by one player of another. Roemer (2001: p. 145) has provided the following description of what Harsanyi’s interpretation means in the specific context of an electoral competition:

> “parties ... can be viewed as playing mixed strategies, as follows. Suppose that each party does not know for certain the *type* of the other party ... . Each party has only a probability distribution over the other party’s type. Each party can compute how the other party will respond to its own policy, for every type that it may be, and this induces a “mixed strategy” - that is, party 1 views party 2 as responding with a probability distribution over strategies, induced by the probability distribution that party 1 has over party 2's type. The appropriate concept of equilibrium, in this case, is a mixed-strategy equilibrium.”

Calvert observed that, when Harsanyi’s interpretation is used, “an equilibrium in mixed-strategies in a game of full information, such as deterministic electoral competition between candidates, is really just a summary of the distribution of outcomes that would be observed if the candidates were slightly uncertain about one another’s perceptions of the plurality function” (p. 41). He added that this using this approach “provides information about a form of candidate uncertainty different from that addressed by the probabilistic voting models” (p. 42).

11.c) Some other solutions that have been proposed

Various other solutions have also been proposed for electoral competitions with deterministic voting. This Subsection briefly describes some of them.

One approach has been to view the set of undominated candidate strategies as a solution set. A variation on this approach has been to take the solution set to be the set of strategies that remains after iterated elimination of (weakly or strongly) undominated candidate strategies. References which have used this type of approach include McKelvey and Ordeshook (1976), Cox (1989) and Ordeshook (1992).

Another approach has been based on the so-called "uncovered set". Saying that alternative x "covers" alternative y means that (i) a majority prefers x to y and (ii) for all $z \in X - \{x, y\}$, if a majority prefers alternative y to z, then there is also a majority that prefers x to z. An alternative is in the "uncovered set" if and only if it is *not* covered by any other alternative. When all majority comparisons are strict (e.g., because there is an odd number of voters and each voter has strict preferences) saying that an alternative y is in the "uncovered set" can (alternatively) be stated as follows: For each other alternative, x, either a) a majority prefers y to x or b) a majority prefers y to a third alternative, z, which a majority prefers to x (that is, a majority prefers y to z *and* a majority prefers z to x). References which have taken the uncovered set to be a solution set for an electoral competition include Miller (1980: pp. 89-93), McKelvey (1986: Section 6), Ordeshook (1986: pp. 184-187) and Mueller (2003: Chapter 11). A variation on this approach has been developed by Schofield (1999: Section 5; 2002: Sections 2 and 3) – using a *superset* of the uncovered set which he calls the "political heart".

Yet another approach has been to take the solution set to be the support of a mixed-strategy equilibrium. References where used this approach has been suggested for an electoral competition with a finite set of alternatives include Laffond, Laslier, and Le Breton (1993, 1994), Laslier (1997) and Dutta and Laslier (1999). With this approach, in a game with a finite set of alternatives, the solution set would consist of all of the alternatives which have a positive probability of being chosen by a candidate. The references listed above established that (in games with a finite sets of alternatives) this solution set is a *subset* of the uncovered set. Subsequently, Banks, Duggan and Le Breton (2002) proved that (under fairly general assumptions) the same conclusion holds for games with infinite sets of alternatives.

The approaches described in the preceding paragraphs all predict that policies will tend to be in a specific, identifiable subset of the society's set of alternatives. However, as was pointed out in Section 6, the models of electoral competition discussed in that section (which have multidimensional sets of alternatives and deterministic voting) rarely have pure-strategy equilibria so the solution concepts discussed in this Subsection all have the following problem: A candidate who confidently selects a policy position using one of these approaches will (usually) subsequently want to change his position - since his opponent will (usually) be able to respond with a position that is preferred by a majority of voters.

12. Conclusion

In this chapter about probabilistic and spatial models of voting, I have viewed the terms "spatial models of voting" and "probabilistic models of voting" as closely related (but not interchangeable) terms.

I have thought about the term "spatial models of voting" in the way that it's used (for instance) in Enelow and Hinich's (1984) book The Spatial Theory of Voting: An Introduction. In addition to using a variation of the term in the title of their book, they also use a variation of the term in the title of Chapter 1 ("Spatial voting models: the behavioral assumptions"). In the first paragraph of that chapter, they state "the spatial theory of voting ... can be traced back as far as the 1920s in the work of Hotelling (1929). The first major works are those of Downs (1957) and Black (1958)" (p. 1). Then, as they proceed through the book, they describe models which are built on the foundation provided by Hotelling, Downs and Black.

I have thought about the term "probabilistic models of voting" in the way that it is used (for instance) in my book Probabilistic Voting Theory (Coughlin (1992)). In addition to using a variation of the term in the title of my book, Chapter 1 described models of electoral competition where "both candidates are ... uncertain about the voters' decisions" – and pointed out that they are commonly "called *probabilistic voting models* (reflecting the fact that the candidates' uncertainty requires a probabilistic description of the voters' choice behavior)" (p. 21).

Since the topic for this chapter is "Probabilistic and Spatial Models of Voting," my goal has been to cover *both* probabilistic voting models *and* other spatial voting models which are closely related to probabilistic voting models – and to state important results about *both* probabilistic voting models *and* other closely related spatial voting models. The other closely related spatial voting models have included Black's model, the Hotelling-Downs model and the subsequent literature on electoral competition where candidates are certain about the decisions that will be made by voters who are not indifferent between the policies embodied by the candidates.

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