The Belief Hierarchy Representation of Harsanyi Type Spaces with Redundancy

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Abstract

In the standard Bayesian formulation of games of incomplete information, some types may represent the same hierarchy of beliefs over the given set of basic uncertainty. Such types are called *redundant types*. Redundant types present an obstacle to the Bayesian analysis because Mertens-Zamir approach of embedding type spaces into the universal type space can only be applied without redundancies. Also, because redundant structures provide different Bayesian equilibrium predictions (Liu [24]), their existence has been an obstacle to the universal argument of Bayesian games. In this paper, we show that every type space, even if it has redundant types, can be embedded into a space of hierarchy of beliefs by adding an appropriate payoff irrelevant parameter space. And, for any type space, the parameter space can be chosen to be the space $\{0, 1\}$. Moreover, Bayesian equilibrium is characterized by this "augmented" hierarchy of beliefs. In this process, we show that the syntactic characterization of types by Sadzik [31] is essentially the same as whether or not they can be mapped to the same "augmented" hierarchy of beliefs. Finally, we show that the *intrinsic correlation* in Brandenburger-Friedenberg [10] can be interpreted as a matter of redundant types and we can obtain their results in our framework.

1 Introduction

One difficulty in dealing with games of incomplete information is the infinite regress of uncertainty. Typically, an agent is uncertain about the payoff functions of the other agents. [1] In order to analyze

[1] The agents' uncertainty about action spaces can be represented as the uncertainty about payoff functions. See Hu-Stuart [23] for the details

an agent's decision under incomplete information, it is not enough to incorporate his belief over the basic uncertainty, that is, the uncertainty about the agents' payoffs. We have to incorporate what the agent believes about what his opponents believe about the basic uncertainty too. And next we have to consider the agent's belief about what his opponents believe about what he believes about the basic uncertainty, and so on *ad infinitum*. Therefore, to deal with games of incomplete information, we have to model this infinite regress of beliefs about beliefs. We call this hierarchy of beliefs a *sequential belief*.

Since Harsanyi [20], we have been dealing with this difficulty by using the notion of *type* and the associated Bayesian game. We postulate that all the informational attributes of agents, including sequential beliefs, can be reduced to one variable called the agent's "type". This postulation allows us to apply equilibrium concepts of games of complete information to games of incomplete information. In this paper, we say that the types defined by Harsanyi are *Harsanyi types* in order to distinguish them from *epistemic types* which we will define later.

Concerning individual informational attributes, we can conceive the information brought by private signals, predetermined personal conjectures (ex. personal characters, or habits in thinking), and so on. We can easily model these attributes with parameters. However, it is not clear that Harsanyi types correctly reflect the agents' sequential beliefs. This suspicion is cleared by Mertens-Zamir [26] and Brandenburger-Dekel [9]. They showed that, under reasonable conditions, the space of the sequential beliefs over the basic uncertainty forms a Harsanyi type space, and we can embed arbitrary Harsanyi type spaces into the space of sequential beliefs. We say that this space of sequential beliefs is the *universal type space* and sequential beliefs are *epistemic types*.

Still we have another difficulty about the sequential beliefs and Harsanyi types. Indeed Mertens-Zamir and Brandenburger-Dekel verified that we can represent sequential beliefs as Harsanyi types, but only when there are no *redundant types*, which are types that are associated with different sequential beliefs. But redundant types ought to be considered, as the following example shows.

Example 1 (Ely-Peski (2006)): Consider the following two Harsanyi type spaces.

Type space A: The payoff parameter space is $S = \{-1, 1\}$, the set of agents is $N = \{1, 2\}$, the set of types is $T_i = \{-1, 1\}$ for $i = 1, 2$, and the belief structure is characterized by a common prior $\mu \in \Delta(S \times T)$ such that

$$\mu(s, t_i, t_{-i}) = \begin{cases} \frac{1}{4} & if\ s = t_i \cdot t_{-i} \\ 0 & otherwise \end{cases}$$

Let $h_i^k(t_i)$ be the kth order belief of the agent i associated with his type t_i. We can derive the sequential beliefs over S in the above structure as follows;

$$h_i^1(-1)[s] = \begin{cases} \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2} & \text{if } s = -1 \end{cases}$$

$$h_i^2(-1)[s] = \begin{cases} \frac{1}{2}h_j^1(-1)[-1] + \frac{1}{2}h_j^1(1)[-1] = \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2}h_j^1(-1)[1] + \frac{1}{2}h_j^1(1)[1] = \frac{1}{2} & \text{if } s = 1 \end{cases}$$

$$h_i^3(-1)[s] = \begin{cases} \frac{1}{2}h_j^2(-1)[-1] + \frac{1}{2}h_j^2(1)[-1] = \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2}h_j^2(-1)[1] + \frac{1}{2}h_j^2(1)[1] = \frac{1}{2} & \text{if } s = 1 \end{cases}$$

$$\vdots$$

The resulting sequential belief of $t_i = -1$ is $\frac{1}{2}$ at each order to the infinite for $i = 1, 2$. In the same way, $h_i(1)$ is $\frac{1}{2}$ at each order for $i = 1, 2$.

Type space B: The payoff parameter is $S = \{-1, 1\}$, the set of agents is $N = \{1, 2\}$, the set of types is $T_i = \{0\}$ for $i = 1, 2$, and the belief structure is characterized by a common prior $\mu \in \Delta(S \times T)$ such that

$$\mu(s, 0, 0) = \begin{cases} \frac{1}{2} & \text{if } s = -1 \\ \frac{1}{2} & \text{if } s = 1 \end{cases}$$

In this case, both agents put probability $\frac{1}{2}$ on each element of S, and this is common knowledge between the agents. Therefore the resulting sequential belief of the type is $\frac{1}{2}, \frac{1}{2}, \ldots$ for $i = 1, 2$. Type space A and type space B have different type structures, but they result in the same sequential beliefs. It means that the representation of a sequential belief using a Harsanyi type is not unique.

Clearly, the type space A and the type space B in the example have different informational structures.[2] In the example, the types $t_i = -1$ and $t'_i = 1$ in the type space A are redundant types. The existence of redundant types shows the difficulty in modeling games of incomplete information. We can also interpret these examples in a different way, that is, when Harsanyi type spaces are given, sequential beliefs over the payoff parameter are not enough to characterize the belief structure of agents. The universal type space does not allow redundancy of types. However, without redundant types, we cannot deal with an interesting class of games such as the type space A. In the type space A, redundancy happens due to the strong correlation of the agents' belief over the payoff parameter and their belief over the other agent's types. Such correlation is common in applications. In Morris-Shin (1996), for instance, the investors share the market information, such as the GDP report and personnel affairs in firms, with some private noises. In their model, the private signals are independent. But, if those private noises are correlated and every agent knows it, in order to model it as a Bayesian game, some types must be strongly correlated with each other and the basic uncertainty so that they result in the same sequential beliefs as in the following example.

Example 2: Correlated public information with noise Let $N = \{1, \cdots, n\}$. There are two states $S = \{G, B\}$. The agents receive private signals X_i about the states from the government. The government tries to hide the state when the state is bad, but it cannot be completely hidden, because there is one agent that receives the true signal. Likewise, when the state is good, the government tries to make it public, but it cannot do so because one agents receives a wrong signal. The

[2] Ely-Peski [17] showed that they have different sets of Bayesian equilibrium and rationalizable strategies.

distribution of the signals and the states are given by a common prior μ such that

$$\forall i \in N,\ \mu(X_1 = G,\ \cdots,\ X_i = B,\ \cdots,\ X_n = G|s = G) = \frac{1}{n},$$
$$\text{Otherwise,}\quad \mu(\ .\ |s = G) = 0.$$
$$\forall i \in N,\ \mu(X_1 = G,\ \cdots,\ X_i = B,\ \cdots,\ X_n = G|s = B) = \frac{1}{n},$$
$$\text{Otherwise,}\quad \mu(\ .\ |s = B) = 0.$$

Then, each type X_i assigns probability $\frac{1}{2}$ to both states. Therefore, the resulting sequential belief is $\frac{1}{2},\ \cdots$ at each type.

The universal type space has received considerable attention lately [3] . But type spaces with redundant types cannot be represented in the universal type space.

In order to make redundant types tractable in the epistemic space, Ely-Peski (2006) constructed a different kind of sequential beliefs called Δ-*hierarchies*. That is, sequential beliefs over the space of probability distributions over the space of parameters. By using beliefs over beliefs as the first order belief, we can deal with the correlation between beliefs over the payoff parameter and beliefs over the other agents' types. And, in particular, some types that would be called redundant under standard sequential beliefs are mapped to different Δ-hierarchies. Δ-hierarchies can represent richer information about the belief structure of the agents than ordinary epistemic types, and give us a better foundation to work on the epistemic analysis of games. In Δ-hierarchies, however, we can only distinguish redundant types up to rationalizable actions. Harsanyi types which have different sets of rationalizable actions result in different Δ-hierarchies, but Harsanyi types which share the same set of rationalizable actions result in the same Δ-hierarchy. In the above example, the types $t_i = 1$ in the type space A and $t_i = 0$ in the type space B can be distinguished from each other in Δ-hierarchies, but $t_i = 1$ and $t_i' = -1$ in the type space A cannot be distinguished there. Therefore we cannot always map Harsanyi type spaces into the space of the Δ-hierarchies isomorphically.

[3] Ex. Weinstein-Yildiz [32], Dekel, *et al* [14, 15], and Bergemann-Morris [6].

Liu [24] took a different approach from Ely-Peski. He augmented the universal type space by adding an additional parameter space, which he called the *payoff irrelevant parameter space.* He showed that any Harsanyi type space, even if it has redundant types, has its isomorphic image in the space of the sequential beliefs over the payoff parameter S and a payoff irrelevant parameter C. However, the payoff irrelevant parameter space that Liu used was the agents' type space T. Therefore the resulting epistemic types space vary depending on Harsanyi type spaces to be studied. Since we cannot compare Harsanyi type spaces on one epistemic space, topological arguments such as Fudenberg-Dekel-Morris [14] and Ely-Peski [18] are not possible here. In this sense, the space that Liu constructed is different from the universal type space that Mertens-Zamir and Brandenburger-Dekel did. Besides, from the epistemic perspective, we cannot obtain any insight into what kind of information beyond the universal type space is needed to deal with the redundancy of types.

In this paper, we offer a solution by finding an *exogenous* payoff irrelevant parameter space. Moreover, for any Harsanyi type spaces to be mapped, the exogenous parameter space can be a two valued set $\{0, 1\}$. To get an intuition of our argument, consider a two person game. Let us make on the agents' Harsanyi type spaces a partition of equivalence classes whose elements have the same sequential belief over the payoff parameter. Equivalently, we sort Harsanyi types into classes of redundant types. In type spaces with redundant types, the beliefs of redundant types have the same probability distribution over the equivalence classes of the other agent's type space although they are different within each equivalence class. This means that even if redundant types have different conjectures over the payoff parameter and the other agents's type, they are different just within each equivalence class of the other agent, not across equivalence classes. Since the members of each equivalence class of the other agent's types cannot be distinguished by their sequential beliefs, the agent's redundant types also result in the same sequential belief. Our method to deal with redundancy is to distinguish the members of each equivalence class by attaching to each type of an agent a different conjecture over a newly added payoff irrelevant parameter. As a result, those redundant types have different first order beliefs over the payoff parameter and the payoff irrelevant parameter. It enables us to distinguish the redundant types of the other agent by their second order beliefs because we can distinguish their different conjectures within each equivalence class. We have a further result when we assume that S and T_i, for all $i \in N$, are uncountable Polish spaces, that is, complete, metrizable and separable spaces.[4] Then it is sufficient to distinguish the redundant types if the payoff irrelevant parameter space has two elements. Here is an explanation. All spaces are infinite and Polish, and so the type space of the agent 1 is Borel equivalent to the closed interval $[0, 1]$. On the other hand, the space of probability measures over $\{0, 1\}$ is homeomorphic to $[0, 1]$. Thus we can assign Borel equivalent different first order beliefs over the set $\{0, 1\}$ to all the types of the player 1. By doing this, we can distinguish the members of each equivalence class of the player 1's redundant types by their first order belief, and so the player 2's redundant types are distinguishable by their second order belief whenever they have different conjectures over the payoff parameter and the player 1's type space.

Now we can completely represent any Harsanyi type space as a subspace of the "universal type space" over $S \times \{0, 1\}$. It is beneficial for two reasons. First, it gives an epistemic foundation of Harsanyi type spaces. Any correlation of beliefs of agents which is not captured by the sequential belief over S can be recovered just by introducing a coin flip as a moderator across agents. Alternatively, any hidden uncertainty in Harsanyi type spaces can be identified as the uncertainty about an agent's personality. For example, whether or not he believes in God. The sequential conjecture over an agent's personality generates the correlation of beliefs over the payoff parameter and agents' types. Second, it allows us to deal with Bayesian games in a "universal" space. The payoff irrelevant parameter $\{0, 1\}$ is exogenous and we do not have to change the payoff irrelevant parameter as in Liu's construction. In fact, as we explain later, the points on $U(S \times C)$ characterizes Bayesian equilibrium.

We have other contributions in this paper. One is to fill the gap between two methods in the epistemic game theory: syntactic one and semantic one, i.e. universal type space approach. Concerning the syntactic method, Sadzik [31] adopted a first order epistemic language *a la* Aumann [5]. He distinguished Harsanyi types with the sets of the sentences which can be true at the types, and showed that this identification of types is essentially equivalent to identifying types with the set of possible Bayesian equilibrium strategies there. Compared to Δ-hierarchies by Ely-Peski, which identifies with IIR, it is a

[4] Even if S and T are countable sets equipped with discrete topology, we can still apply the following argument since we can embed T to $[0, 1]$ Borel isomorphically.

finer epistemic characterization of Harsanyi types. And, this result also gives an epistemic characterization of Bayesian equilibrium, which is the most successful one so far. However his method is totally different from the existing literature, and we could not compare Sadzik's syntactic characterization with the other universal type space approach such as Liu's. In this paper, we show that Sadzik's syntactic characterization of types is essentially equivalent to whether or not they are mapped to the same sequential beliefs on $U(S \times C)$.

Another contribution is in the Bayesian formulation of complete information games. Since Aumann [2, 3], mixed strategies and correlated strategies in complete information games have been given a foundation by assuming a basic uncertainty not described in the game [5] and reinterpreting them as incomplete information games. However, the resulting games often have redundant types. Brandenburger-Friedenberg [10] considered the set of correlated equilibria which can be achieved only thorough the correlation of sequential beliefs over the basic uncertainty(*intrinsic correlation*). This is equivalent to consider the set of correlated equilibria achieved in Bayesian formulations without redundant types. Therefore, we can apply our result and show that every correlated equilibrium can be achieved through intrinsic correlation when we add a coin flip to the basic uncertainty. It is the same as the result in Brandenburger-Friedenberg derived in a different way.

This paper is organized as follows. In Section 2 and 3, we present the formal model and the proof of our main result: the elimination of redundancy by adding the payoff irrelevant parameter space $C = \{0, 1\}$. In Section 4, we characterize our result with Bayesian equilibrium, and interpret syntactic approach on the universal type space. In Section 5, we discuss the intrinsic correlation in terms of redundant types. In the Appendix, we give detailed proofs about some measurability issues involved in our construction.

2 Preliminaries

Let X be an arbitrary set. We use $\Delta(X)$ to denote the space of the probability measures over X. When X is equipped with a topology, we use $\Sigma(X)$ to denote the Borel σ-algebra on X.

[5] Without loss of generality, we can consider it to be the space of actions.

Let N be a finite set, and $(Y_i)_{i \in N}$ be a family of sets. Then, for any $i \in N$, we use Y_{-i} to denote the product space $\Pi_{j \in N \setminus \{i\}} Y_j$.

2.1 Harsanyi type space

We consider a finite set of agents $N = \{1, \ldots n\}$. All the agents face the same basic uncertainty about their payoffs. It can be represented by a parameter space S.[6] We call this S the *payoff parameter space*. A *Harsanyi type space* is a tuple $\langle S, (T_i)_{i \in N}, (\lambda_i)_{i \in N} \rangle$, where, for each $i \in N$, λ_i is a function from T_i to $\Delta(S \times T_{-i})$. We call each element $t_i \in T_i$ a *Harsanyi type*. By the function λ_i, each type stands for a belief over the payoff parameter and the other players' types. Hereafter we make some assumptions on Harsanyi type spaces.

Assumption 1: The parameter space S and the each agent's type space T_i are uncountable Polish spaces.

Let $T \equiv \Pi_{i \in N} T_i$. Then, as it is known, the product type space T is also a Polish space.

In many works such as Mertens-Zamir [26] *etc.*, the belief mapping λ_i is assumed to be homeomorphism. Here we relax this usual assumption slightly.

Definition 2.1. *A function $f : X \to Y$ is bimeasurable if f is measurable and, for each measurable set $E \subset X$, $f(E)$ is also measurable.*

Assumption 2: For each $i \in N$, the function λ_i is a *bimeasurable* injection.

This assumption precludes *purely redundant* types, which are Harsanyi types $t_i, t_i' \in T_i$ such that $t_i \neq t_i'$ and $\lambda_i(t_i) = \lambda_i(t_i')$.

[6] See Mertens-Zamir [26], and Hu-Stuart [23].

2.2 Universal type spaces

The universal type space was introduced by Mertens-Zamir[26]. It is the space of the sequential beliefs over S which satisfy some coherency conditions. They showed that the space is also a Harsanyi type space and any Harsanyi type space without redundant types is embedded there. To define the universal type space, we have to define the space of the sequential beliefs first. Let a family of spaces $(Z^k)_{k\geq 1}$ be such that

$$Z^1 \equiv S$$
$$For\ k > 1,\ Z^k \equiv Z^{k-1} \times \Delta(Z^{k-1}).$$

The space Z^k is the set of the kth order beliefs over S. We say that $\Pi_{k=1}^{\infty} Z^k$ is the *sequential belief space* and each element of it is the *sequential belief*. Let a sequential belief be $z \equiv (z_1, \cdots)$ where, for all $k \in \mathbb{N}$, $z_k \in Z_k$. We say that z satisfies *coherency* if, for all $k \in \mathbb{N}$, the marginal distribution of z_{k+1} over Z_k is the same as z_k. Under coherency of beliefs, we can consider each element in $\Pi_{k=1}^{\infty} Z^k$ as a projection limit. Let the set of the projection limits be Z^∞. We say that each $e_i \in Z^\infty$ is an *epistemic type*. The *universal type space* is the set of all the sequential beliefs that satisfy coherency. We denote it as $U(S)$. Mertens-Zamir showed the following strong theorem about the universal type space.

Theorem 2.2. *(Mertens-Zamir [26]) The universal type space $U(S)$ and its associated natural homeomorphism constitutes a Harsanyi type space.*

Then we can define the function which maps Harsanyi types onto the sequential belief space. Let the first order mapping $h_i^1 : T_i \to \Delta(S)$ be such that

$$h_i^1(t_i) = Marg_{(S)}\lambda_i(t_i).$$

For $k > 1$, let the kth order mapping $h_i^k : T_i \to \Delta(Z^k)$ be such that

$$h_i^k(t_i) = \lambda_i(t_i) \circ [Id_S, (h_j^{k-1})_{j \in N \setminus \{i\}}]^{-1},$$

where Id_S is an identical function from S to S.

We say that the function $(h_i^k)_{k=1}^{\infty} : T_i \to \Pi_{k=1}^{\infty} Z^k$ is the *hierarchy mapping*. Let $h \equiv (h_i)_{i \in N}$. Then, this h enables us to map any Harsanyi type space to the sequential belief space. Also you can see that sequential beliefs derived in this way satisfy the coherency condition.

3 An extended sequential belief space

In this section, we extend the universal type space by adding a payoff-irrelvant parameter space C. And we show that we can isomorphically embed Harsanyi type spaces there even if they have redundant types.

3.1 Redundant types

Let $\Lambda = \langle S, T, (\lambda)_{i \in N} \rangle$ be a Harsanyi type space. Mertens-Zamir showed that Harsanyi type spaces can be embedded as a subspace of $U(S)$ homeomorphically only if they have no redundant types. To discuss the matter, we have to define redundant types first.

Definition 3.1. *In a Harsanyi type space Λ, two Harsanyi types t_i and $t_i' \in T_i$ are redundant if $h_i(t_i) = h_i(t_i')$.*

We say that the Harsanyi types which are not redundant are *non-redundant types*. Then we can formally state what Mertens-Zamir showed.

Proposition 3.2. *(Mertens-Zamir [26]) Any Harsanyi type space without redundant types can be embedded onto $U(S)$ homeomorphically. And the hierarchy mapping h is the unique embedding.*

3.2 Extension with a payoff irrelevant parameter space

Now we construct an extended space of sequential beliefs so that we can embed Harsanyi type spaces there even if they have redundant types. We introduce a parameter space $C = \{0, 1\}$ and consider the sequential belief space over $S \times C$ instead of S. In the rest of this section, we assume that $N = \{1, 2\}$.

Let $C \equiv \{0, 1\}$. We assume that any element does not affect the payoffs of the agents. Therefore we call C the *payoff irrelevant parameter space*. We define sequential beliefs over $S \times C$ and construct the coherent sequential belief space over $U(S \times C)$ in the same way as we did over S.

Let

$$Z_1 \equiv S \times C,$$
$$\forall k \geq 2,\ Z_k \equiv \Delta(\Pi_{n=1}^{n=k-1} Z_n).$$

And let

$$\begin{aligned} H^k(S \times C) &\equiv \Delta(\Pi_{n=1}^{n=k} Z_k) \\ &= Z_{k+1}. \end{aligned}$$

and

$$\begin{aligned} H(S \times C) &\equiv \Pi_{k=1}^{k=\infty} H^k(S \times C) \\ &= \Pi_{k=1}^{k=\infty} \Delta(Z_k). \end{aligned}$$

For each k, $H^k(S \times C)$ is the set of the kth order belief over $S \times C$. Let $U(S \times C) \subset \Pi_{i \in N} H(S \times C)$ be the product space of the coherent sequential beliefs.

We also define Harsanyi type spaces based on $S \times C$ by the sequence $\Phi = \langle S \times C, V, (\phi_i)_{i \in N} \rangle$ where ϕ_i is a bimeasurable injection from V_i to $\Delta(S \times C \times V_{-i})$.

Before we embed a Harsanyi type space onto $U(S \times C)$, we extend it to a Harsanyi type space on $S \times C$. To do that, we should clarify what is "isomorphism" between Harsanyi type spaces.

Definition 3.3. *(Liu [24]) Let $X = \langle S, T, \lambda \rangle$ and $Y = \langle S \times C, V, \phi \rangle$ be Harsanyi type spaces on S and $S \times C$ respectively. Then, X and Y are* S-isomorphic *to each other if there exists a $g = (g_i)_{i \in \{0\} \cup N}$ such that (1) $g_0 : S \to S$ is an identity function, (2) $g_i : T_i \to V_i$ is Borel equivalence for all $i \in N$, and (3) $Marg_{S \times V} \phi_i(v_i) = \lambda_i(t_i) \circ g^{-1} \circ Proj_{S \times V}$.*

Hereafter, when Harsanyi type spaces X and Y are S-isomorphic, we use $X \sim_S Y$. And when both spaces are defined on S, we use $X \sim Y$.

Next, we want to construct a Harsanyi type space on $S \times C$ which is S-isomorphic to the original type space on S. For the construction, we need the next well-known theorem.[7]

Theorem 3.4. *Let X be an uncountable Polish space. Then X is Borel equivalent to the closed interval* $[0, 1]$.

Let $\Lambda \equiv \langle S, (T_i)_{i \in \{1,2\}}, (\lambda_i)_{i \in \{1,2\}} \rangle$ be a Harsanyi type space. Since T_i is an uncountable Polish space, there exists a Borel equivalence from T_i to $[0, 1]$. Let this equivalence be $p_i : T_i \to [0, 1]$. Using p_i, we define a Harsanyi type space $\Phi = \langle S \times C, (V_i)_{i \in \{1,2\}}, (\phi_i)_{i \in N} \rangle$ so that

[7] See Royden [30] for the detailed argument.

1. For all $i \in \{1,2\}$, $V_i = [0,1]$.

2. For all $i \in \{1,2\}$, $\phi_i : V_i \to \Delta(S \times C \times V_{-i})$ satisfies the next property;

   For the agent 1,

$$\forall v_1 \in V_1, \qquad Marg_{(S\times V_2)}\phi_1(v_1) = \lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1},$$
$$\forall E \in \Sigma(S \times V_2), \qquad \phi_1(v_1)[E \times \{0\}] = v_1\lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1}[E].$$

   For the agent 2,

$$\forall v_2 \in V_2, \qquad Marg_{(S\times V_1)}\phi_2[v_2] = \lambda_2[p_2^{-1}(v_2)] \circ [id_S,\ p_1]^{-1},$$
$$Marg_{(C)}\phi_2(\{0\}) = 1.$$

The bimeasurability of $(\phi_i)_{i\in\{1,2\}}$ is proven in the appendix. Then, you can see that Φ is a well defined Harsanyi type space. Concerning this Harsanyi type space Φ, we have the next fundamental lemma.

Lemma 3.5. *The above type space* Φ *is* S-isomorphic *to* Λ.

Proof. Let $Id_S : S \to S$ be identity function. Then, (Id_S, p_1, p_2) is S-isomorphism from Λ to Φ by construction. □

3.3 S-isomorphic embedding onto $U(S \times C)$

We go to the main part of this paper. We show that, in the Harsanyi type space Φ defined above, all elements of V_i correspond to different sequential beliefs over $S \times C$.

Theorem 3.6. *Let Λ and Φ be Harsanyi type spaces defined above. Then, for each $i \in \{1, 2\}$, the agent i's hierarchy mapping induced by Φ, $h_i : V_i \to H(S \times C)$, is an injection.*

Proof. Let $h_i^k : V_i \to H_i^k(S \times C)$ be the agent i's kth order belief mapping on $S \times C$ induced by Φ, and let $g_i^k : T_i \to H_i^k(S)$ be the agent i's kth order belief mapping onto S induced by Λ.

(Step 1: For the agent 1)

Let $v_1, v_1' \in V_i$ be such that $v_1 \neq v_1'$. His first order belief of v_1 is

$$\begin{aligned}\forall E \in \Sigma(S),\ h_1^1(v_1)[E \times \{0\}] &= \phi_1(v_1)[E \times \{0\} \times V_2] \\ &= v_1 \lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1}[E \times V_2] \\ &= v_1 g_1^1(p_1^{-1}(v_1))[E].\end{aligned}$$

By the symmetric argument,

$$\forall E \in \Sigma(S),\ h_1^1(v_1')[E \times \{0\}] = v_1' g_1^1(p_1^{-1}(v_1'))[E].$$

(Case 1:) Suppose that $v_1 g_1^1[p_1^{-1}(v_1)](E) = v_1' g_1^1[p_1^{-1}(v_1')](E)$. Then, since $v_1 \neq v_1'$, $g_1^1[p_1^{-1}(v_1)](E) \neq g_1^1[p_1^{-1}(v_1')](E)$.

On the other hand,

$$\begin{aligned}h_1^1(v_1)[E \times C] &= \phi_1(v_1)[E \times C \times V_2] \\ &= Marg_{(S \times V_2)} \phi_1(v_1)[E \times V_2] \\ &= \lambda_1(p_1^{-1}(v_1)) \circ [id_S,\ p_2]^{-1}[E \times V_2] \\ &= g_1^1(p_1^{-1}(v_1))[E].\end{aligned}$$

From these results, we have

$$\begin{aligned} h_1^1(v_1)[E \times \{1\}] &= h_1^1(v_1)[E \times C] - h_1^1(v_1)[E \times \{0\}] \\ &= g_1^1(p_1^{-1}(v_1))[E] - v_1 g_1^1(p_1^{-1}(v_1))[E] \\ &= g_1^1(p_1^{-1}(v_1))[E] - v_1' g_1^1(p_1^{-1}(v_1'))[E] \\ &\neq g_1^1(p_1^{-1}(v_1'))[E] - v_1' g_1^1(p_1^{-1}(v_1'))[E] \\ &= h_1^1(v_1')[E \times \{1\}]. \end{aligned}$$

Thus h_1 is injective.

(Case 2:) Suppose that $v_1 g_1^1[p_1^{-1}(v_1)](E) \neq v_1' g_1^1[p_1^{-1}(v_1')](E)$. It means that $h_1^1(v_1)[E \times \{0\}] \neq h_1^1(v_1')[E \times \{0\}]$. Thus h_1 is injection.

(Step 2: For the agent 2)

Let $v_2, v_2' \in V_2$ be such that $v_2 \neq v_2'$. Concerning his first order belief, by construction,

$$\begin{aligned} &\forall E \in \Sigma(S), \\ &\quad h_2^1(v_2)[E \times \{0\}] = g_2^1(p_2^{-1}(v_2))[E]. \\ &\quad h_2^1(v_2)[E \times \{1\}] = 0. \end{aligned}$$

Let, for each $\mu_1 \in \Delta(S \times C)$, $h_1^{-1}(\mu_1) \equiv \{v_1 \in V_1 : h_1^1(v_1) = \mu_1\}$. As we have shown, the function $h_1^1 : V_1 \to \Delta(S \times C)$ is injective. Therefore $h_1^{-1} : h_1^1(V_1) \to V_1$ is the well defined inverse bijection.

Then we can derive the agent 2's second order belief over $S \times C$.[8]

[8]Concerning the bimeasurablity of h_1^1, see appendix.

Note that

$$\begin{aligned}
&\forall E \in \Sigma(S),\ \forall Q \in \Sigma(\Delta(S \times C)) \\
&\qquad h_2^2[v_2](E \times \{0\} \times Q) = \phi_2(v_2)[E \times \{0\} \times h_1^{-1}(Q)] \\
&\qquad\qquad = \lambda_2(p_2^{-1}(v_2)) \circ [id_S,\ p_1]^{-1}[E \times h_1^{-1}(Q)].
\end{aligned}$$

Since $\lambda_2 : T_2 \to \Delta(S \times T_1)$ is a bimeasurable injection, $\lambda_2(p_2^{-1}(v_2)) \neq \lambda_2(p_2^{-1}(v_2'))$. By Dynkin's lemma[9], there exists a rectangle $F \equiv \hat{S} \times \hat{T}_1$ such that $\hat{S} \in \Sigma(S)$, $\hat{T}_1 \in \Sigma(T_1)$, and $\lambda_2(p_2^{-1}(v_2))[F] \neq \lambda_2(p_2^{-1}(v_2'))[F]$. Let $\hat{V}_1 \equiv p_1(\hat{T}_1)$. Then, $\hat{V}_1 \in \Sigma(V_1)$ and $h_1^1(\hat{V}_1) \in \Sigma(\Delta(S \times C))$. Therefore

$$\begin{aligned}
h_2^2(v_2)(\hat{S} \times \{0\} \times h_1^1(\hat{V}_1)) &= \phi_2(v_2)[\hat{S} \times \{0\} \times h_1^{-1}(h_1^1(\hat{V}_1))] \\
&= \phi_2(v_2)[\hat{S} \times \{0\} \times \hat{V}_1] \\
&= \lambda_2[p_2^{-1}(v_2)] \circ [id_S, p_1^{-1}](\hat{S} \times \hat{V}_1) \\
&= \lambda_2(p_2^{-1}(v_2))(\hat{S} \times \hat{T}_1) = \lambda_2(p_2^{-1}(v_2))[F] \\
&\neq \lambda_2(p_2^{-1}(v_2'))[F] \\
&= h_2^2(v_2')[\hat{S} \times \{0\} \times h_1^1(\hat{V}_1)].
\end{aligned}$$

It means that $h_2(v_2) \neq h_2(v_2')$. Therefore, h_2 is injection. $\square$

So far we did not consider topological structures of Harsanyi type spaces except that they are Polish. As it plays a crucial role in Weistein-Yildiz [32] and others, it is important to show that each agent's type space V_i is homeomorphic to the belief space $\Delta(S \times C \times V_{-i})$.

Definition 3.7. *A Harsanyi type space $\mathcal{X} = \langle X, T, (x_i)_{i \in N} \rangle$ is a* continuous type space *if, for all $i \in N$, $x_i : T_i \to \Delta(X \times T_{-i})$ is homeomorphic embedding.*

Next we show that the embedded image on $U(S \times C)$ of each Harsanyi type by the hierarchy mapping

[9] See Theorem 10-10 in [1]

is a continuous Harsanyi type space.

Lemma 3.8. *Let Φ be a type space and $H(S \times C)$ be the space of sequential belief over $S \times C$ as we defined before. The function $h_i : V_i \to H(S \times C)$ is the full hierarchy mapping. Now let $f : h_i(V_i) \to \Delta(S \times C \times h_i(V_{-i}))$ be such that $f(h_i(v_i)) \equiv \phi(v_i) \circ [id_{(S\times C)}, h_{-i}]^{-1}$. Then, f is homeomorphism.*

Proof. Since $S \times C$ is a Polish space, there exists a unique homeomorphism $\psi : H(S \times C) \to \Delta(S\times C\times H(S\times C))$ such that, for each $m \in H(S\times C)$, $\psi(m)$ is the Kolmogorov extension of m.[10]. So it is enough to show that, for all $i \in \{1, 2\}$ and $v_i \in V_i$, $f_i(h_i(v_i))$ is the Kolmogorov extension of $h_i(v_i)$

Let $m_i \in h_i(V_i)$. First, for all $E \in \Sigma(S \times C \times H(S \times C))$, by letting $f_i(m_i)(E) \equiv f_i(m_i)[E \bigcap (S \times C \times h_{-i}(V_{-i}))]$, we can extend $f_i(m_i)$ so that $f_i(m_i) \in \Delta(S \times C \times H(S \times C))$. And as we defined before,

$$Z_1 \equiv S \times C,$$
$$\forall k \geq 2,\ Z_k \equiv \Delta(\Pi_{n=1}^{n=k-1} Z_n),$$

$$\begin{aligned} H^k(S \times C) &= \Delta(\Pi_{n=1}^{n=k} Z_k) \\ &= Z_{k+1}, \end{aligned}$$

and

$$\begin{aligned} H(S \times C) &= \Pi_{k=1}^{k=\infty} H^k(S \times C) \\ &= \Pi_{k=1}^{k=\infty} \Delta(Z_k). \end{aligned}$$

[10]See Prop 1 and Prop 2 in Brandenburger-Dekel [9]

The equations above also imply that

$$H(S \times C) = \Pi_{n=2}^{n=\infty} Z_n.$$

Therefore,

$$S \times C \times H(S \times C) = \Pi_{n=1}^{n=\infty} Z_n.$$

From these equalities, we have $f_i(m_i) \in \Delta(\Pi_{n=1}^{n=\infty} Z_n)$, and $m_i \in H(S \times C) = \Pi_{k=1}^{k=\infty} \Delta(Z_k)$.

To show that $f_i(m_i)$ is the Kolmogorov extension of m_i, it is enough to show that the next property holds:

$$\forall k,\ Marg_{(\Pi_{n=1}^{n=k} Z_n)} f_i(m_i) = Proj_k m_i.$$

Let $E \in \Sigma(\Pi_{n=1}^{n=k} Z_n)$ and $\hat{E} = E \times \Pi_{n=k+1}^{n=\infty} Z_n$. Then,

$$\begin{aligned} f(m_i)(\hat{E}) &= \phi[v_i] \circ [id_{(S \times C)}, h_{-i}]^{-1}(\hat{E} \bigcap h_{-i}(V_{-i})) \\ &= \phi(v_i)(E_1 \times \hat{V}_{-i}^k), \\ where\ \ \hat{V}_{-i}^k &= \{v_{-i} \in V_{-i} : h_{-i}^k(v_{-i}) \in \Pi_{n=2}^{n=k} E_n\}. \end{aligned}$$

On the other hand, from the kth order belief of the agent i,

$$\begin{aligned} \exists v_i \in V_i,\ Proj_k m_i[E] &= h_i^k[v_i](E) \\ &= \phi_i[v_i](E_1 \times \hat{V}_{-i}). \end{aligned}$$

This equation means that $f_i(m_i)[\hat{E}] = Proj_k(m_i)(E)$. Consequently, $f_i(m_i)$ is the Kolmogorov extension of m_i. □

As a consequence, we have the next theorem.

Theorem 3.9. *For any Harsanyi type space $\Lambda = \langle S, T, (\lambda_i)_{i\in\{1,2\}}\rangle$, there exists a* continuous BL-subspace *in $U(S \times C)$ which is S-isomorphic to Λ.*

Proof. Let a Harsanyi type space $\Phi = \langle S \times C, V, \phi\rangle$ be an S-isomorphic extension of Λ, and let $E_i = h_i(V_i)$ for all $i \in N$. Let $\mathcal{E} = \langle S \times C, E, (f_i)_{i\in N}\rangle$, where f_i is defined as in the lemma. Since h_i is bimeasurable injection, $\mathcal{E}$ is S-isomorphic to Φ by construction. As a direct result of the lemma, $\mathcal{E}$ is a continuous Harsanyi type space. □

3.4 Extension to $N > 2$

We can extend the above theorems to N-person game. Let N be the finite set of the agents and $|N| = n$. Consider an N-person Harsanyi type space $\Lambda \equiv \langle S, T, (\lambda_i)_{i\in N}\rangle$ as before. We maintain the same assumptions on S, T, C and λ_i.

We can define an extension of Λ on $S \times C$, $\Phi = \langle S \times C, V, (\phi_i)_{i\in N}\rangle$, as follows. For all $i \in N$, let $p_i : T_i \to [0, 1]$ be a Borel equivalence. Let Φ be such that

$$\begin{aligned}
&\forall i \in N, \quad V_i = [0, 1].\\
&\forall i \in N\backslash\{1\},\ \forall v_i \in V_i,\ \forall E \in \Sigma(S \times V_{-i}),\\
&\qquad \phi_i(v_i)[E \times \{0\}] = v_i\{\lambda_i(p_i^{-1}(v_i)) \circ [id_s, p_{-i}^{-1}](E)\}.\\
&\qquad Marg_(S \times V_{-i})\phi_i(v_i) = \lambda_i(p_i^{-1}(v_i)) \circ [id_s, p_{-i}^{-1}].
\end{aligned}$$

And,

$$\begin{aligned}
&\forall v_1 \in V_1,\ \forall E \in \Sigma(S \times V_{-1}),\\
&\qquad \phi_1(v_1)[E \times \{0\}] = \lambda_1(p_1^{-1}(v_1)) \circ [id_s, p_{-1}^{-1}](E).\\
&\qquad Marg_(S \times V_{-1})\phi_1(v_1) = \lambda_1(p_1^{-1}(v_1)) \circ [id_s, p_{-1}^{-1}].
\end{aligned}$$

In the same way as we did above, we can show that Φ is S-isomorphic to Λ and the resulting hierarchy mapping is injection.

4 The characterization of the sequential belief in $U(S \times C)$

In this section, we show that the sequential belief in $U(S \times C)$ and Bayesian equilibrium characterize each other. In this process, a new notion, *symmetric types*, gives us a new insight and help us to establish the characterization.

4.1 Symmetric types

Throughout this section, we assume that S and T are finite for technical convenience. Unless otherwise stated, the results below are valid in the general case of infinite S and T.

Definition 4.1. *Let $t_i, t_i' \in T_i$ in Λ. The Harsanyi types t_i and t_i' are one sided symmetric if there exists a permutation $\pi_{-i} : T_{-i} \to T_{-i}$ such that, for all $E \in \Sigma(S)$,*

$$\forall t_{-i} \in T_{-i}, \quad \lambda_i(t_i)(s, t_{-i}) = \lambda_i(t_i')(s, \{\pi_{-i}(t_{-i})\}).$$

Definition 4.2. *The Harsanyi types t_i and t_i' in Λ are symmetric if (1) t_i and t_i' are one side symmetric, (2) for each $t_{-i} \in T_{-i}$, t_{-i} and $\pi_{-i}(t_{-i})$ are one sided symmetric with regard to a permutation $\pi_i : T_i \to T_i$, and (3) $t_i' = \pi_i(t_i)$.*

The permutation $\pi \equiv (\pi_i, \pi_{-i})$ is just a renaming of types in Λ. The above definition states that we can exchange the roles of symmetric types without changing the entire structure of Λ. As a result we can say that symmetric types have the same set of Bayesian equilibria in any game.

Definition 4.3. *A game Γ on S is a tuple of $((u_i)_{i \in N}, A)$, where $u_i : A \times S \to \mathbb{R}$.*

Let $\beta_i : T_i \to A$ be the agent i's (pure) strategy. Bayesian equilibrium is defined as follows;

Definition 4.4. *A tuple of strategies* $\beta \equiv (\beta_i)_{i \in N}$ *is Bayesian equilibrium if, for all* $i \in N$, $t_i \in T_i$, *and* $a_i \in A_i$, $\int_{S \times T_{-i}} u_i(\beta_i(t_i), \beta_{-i}, s) d\lambda_i(t_i) \geq \int_{S \times T_{-i}} u_i(a_i, \beta_{-i}, s) d\lambda_i(t_i)$.

Definition 4.5. *For each* $t \in T$ *in* Λ *and* Γ,

$$BE(t, \Gamma) \equiv \{a \in A : \exists \beta^* \text{ s.t. } \beta^* \text{ is Bayesian equilibrium in } \Gamma, \text{ and } \beta^*(t) = a\}.$$

Proposition 4.6. *Let* $t_i, t'_i \in T_i$ *be symmetric types. Then, for any game* Γ, $BE(t_i, \Gamma) = BE(t'_i, \Gamma)$.

Proof. Let $\tilde{t}_i, \hat{t}_i \in T_i$ be symmetric types. Suppose that $a^*_i \in BE(\tilde{t}_i, \Gamma)$. Then, there exists a B.E. $\tilde{\beta}$ such that $\tilde{\beta}_i(\tilde{t}_i) = a^*_i$.

Now $\tilde{t}_i$ and $\hat{t}_i$ are symmetric. Therefore , there exists $(\pi_i)_[i \in N$ such that

$$\forall i \in N,\ \forall t_i \in T_i,\ \lambda_i(t_i) = \lambda_i(\pi_i(t_i)) \circ [Id_S, \pi_{-i}]. \tag{1}$$

Let $\hat{\beta}$ be a pair of strategies such that, for each $i \in N$ and $t_i \in T_i$, $\tilde{\beta}_i(t_i) = \hat{\beta}_i(\pi_i(t_i))$. Under the strategy $\hat{\beta}$, for each $i \in N$ and $t_i \in T_i$, the expected payoff to the agent i by taking an action a_i at his type $\pi_i(t_i)$ is:

$$\begin{aligned} U_i(a_i, \hat{\beta}_{-i}, \pi_i(t_i)) &= \int u_i(s, a_i, \hat{\beta}_{-i}(t_{-i})) d\lambda_i(\pi_i(t_i)) && (2) \\ &= \int u_i(s, a_i, \tilde{\beta}_{-i}(\pi_{-i}(t_{-i}))) d\lambda_i(\pi_i(t_i)) && (3) \\ &= \int u_i(s, a_i, \tilde{\beta}_{-i}(t_{-i})) d\lambda_i(t_i) && (4) \\ &= U_i(a_i, \tilde{\beta}_{-i}, t_i). && (5) \end{aligned}$$

Since $\tilde{\beta}_i(t_i) \in ArgmaxU_i(a_i, \tilde{\beta}_{-i}, t_i)$, we have, for each $i \in N$ and $t_i \in T_i$, $\hat{\beta}_i(\pi_i(t_i)) \in ArgmaxU_i(a_i, \hat{\beta}_{-i}, \pi_i(t_i))$.

Therefore, $\hat{\beta}$ is also BNE of the game Γ. Since $\hat{t}_i = \pi_i(\tilde{t}_i)$, $\hat{\beta}_i(\hat{t}_i) = \tilde{\beta}_i(\tilde{t}_i)$. Therefore, $BE(\tilde{t}_i, \Gamma) \subset BE(\hat{t}_i, \Gamma)$. By the symmetric argument, we also have $BE(\tilde{t}_i, \Gamma) \supset BE(\hat{t}_i, \Gamma)$. Thus $BE(\tilde{t}_i, \Gamma) = BE(\hat{t}_i, \Gamma)$. □

We show that symmetric types characterize the sequential belief in $U(S \times C)$.

Proposition 4.7. *Symmetric types are S-isomorphically mapped to the same points on $U(S \times C)$.*

Proof. Let $\tilde{t}_i, \hat{t}_i \in T_i$ be symmetric types. For $i = 1, 2$, let $\pi \equiv (\pi_i)_{i \in N}$ be a permutation defined in the definition of symmetric types.

We want to show that π is a S-isomorphism from Λ to itself. Since T is countable, π_i is Borel isomorphism from T_i to T_i. By the definition of symmetry, we have that;

$$\forall i \in N,\ \forall t_i \in T_i,\ \lambda_i(t_i) = \lambda_i(\pi_i(t_i)) \circ [Id_S, \pi_{-i}].$$

It means that π is S-isomorphism from Λ to Λ. Also, the definition of symmetry implies that $\hat{t}_i = \pi_i(\tilde{t}_i)$. Both of the types are S-isomorphically mapped to each other. Thus they are S-isomorphically mapped to the same points on $U(S \times C)$. □

Next we have to show that the symmetry characterizes the sequential belief in $U(S \times C)$.

Proposition 4.8. *When we embed Λ to $M \subset U(S \times C)$, if two Harsanyi types in T_i can be mapped to the same point in M by some S-isomorphisms, then they are symmetric.*

Proof. Let $t_i, t'_i \in T$ and h be an S-isomorphic embedding from T to M. Suppose that there exists another S-isomorphism $h' : T \to M$ such that $h'(t') = h(t)$. Then $g \equiv h^{-i} \circ h'$ is an S-isomorphism from T to T and $t = g(t')$. By the definition, t and t' are symmetric with regard to the permutation g. □

4.2 Characterization of symmetric types

Next we show that symmetric types are characterized by Bayesian equilibrium. We will make use of a result of Sadzik [31], who adopted the syntactic approach to obtain an epistemic characterization of Bayesian equilibrium. We show that Sadzik's syntactic condition is characterized by the symmetry of types.

We do not deal with the syntactic details here. We introduce only the part of the paper that we need here. Sadzik added *signal* to the Harsanyi type.

Definition 4.9. *For each $i \in N$, let $X_i \equiv \{0,1\}^{\mathbb{N}}$. A signal from T_i to X_i is a function $z_i : T_i \to X_i$.*

Let $z \equiv (z_i)_{i \in N}$. We denote Z as the set of all possible z. We assume z is common knowledge among the agents. Then, given a Harsanyi type space Λ and the realized private information z, we can derive a hierarchy mapping $\delta_i^z : T_i \to U(S \times X)$ in the same way we derived sequential beliefs over S. Let $\delta^z \equiv (\delta_i^z)_{i \in N}$. Notice that z does not have to be a bijection. Therefore, the image of Λ by δ^z is no longer S-isomorphic to Λ generally.

We assume that A is Polish. Since X_i is the Hilbert cube, we can embed A_i to X_i. Therefore we can interpret the set of signals Z as the set of potential strategies. For the characterization, we need the next notation.

Definition 4.10. *For each $t \in T$ in Λ and Γ,*

$$LBE(t, \Gamma) \equiv \{a \in A : \exists \beta^* \text{ s.t. } \beta^* \text{ is B.E. in } \Lambda^t \text{ and } \Gamma, \text{ and } \beta^*(t) = a\}.$$

Here Λ^t is the smallest sub type space of Λ which includes t.

Theorem 4.11. *(Sadzik [31]) For $t, t' \in T$ in Λ, if $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$, then, $LBE(t, \Gamma) = LBE(t', \Gamma)$ for any Γ.*

Theorem 4.12. *(Sadzik [31]) For $t, t' \in T$ in Λ, if $BE(t, \Gamma) = BE(t', \Gamma)$ for any Γ, then $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$.*[11]

[11] When T is infinite, the latter part is relaxed to the equality of the closure of $\{\delta^z(t) : z \in Z\}$ and $\{\delta^z(t') : z \in Z\}$.

Next we show how to interpret Sadzik's characterization by using the universal type space argument.

Proposition 4.13. *For $t, t' \in T$ in Λ, if t and t' are symmetric, then $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$.*

Proof. Suppose that $t, t' \in T$ are symmetric and π is the associated permutation on T such that $t = \pi(t')$. Let $z \in Z$. Then $z \circ \pi^{-i} \in Z$. All we have to show is $\delta^z(t') = \delta^{z\circ\pi^{-i}}(t)$.

Let δ^z_k be the kth order belief mapping induced by a signal z, and let H^k_X be the space of the kth order sequential beliefs over $(S \times X)$. Then

$$\forall i \in N,\ \forall l \in T,$$

$$\forall s \in S,\ \forall x \in X,\ \delta^z_{1,i}(l_i)(s,x) = \lambda_i(l_i)[Id_S, z_i]^{-1}(s,x).$$

$$\forall E \in \Sigma(H^k_X),\ \delta^z_{k+1,i}(l_i)[E] = \lambda_i(l_i)[Id_S, \delta^z_{k,-i}]^{-1}[E].$$

We also have

$$\forall i \in N,\ \forall l \in T,$$

$$\forall s \in S,\ \forall x \in X,\ \delta^{z\circ\pi}_{1,i}(l_i)(s,x) = \lambda_i(l_i)[Id_S, z_{-i} \circ \pi^{-i}_{-i}]^{-1}(s,x).$$

$$\forall E \in \Sigma(H^k_X),\ \delta^{z\circ\pi}_{k+1,i}(l_i)[E] = \lambda_i(_i)[Id_S, \delta^{z\circ\pi^{-i}}_{k,-i}]^{-1}[E].$$

we show that $\delta^z(t') = \delta^{z\circ\pi^{-i}}(t)$ by mathematical induction with regard to k. For $k = 1$, we have , for all $i \in N$, and any $t, t' \in T$ such that $t = \pi(t')$, the equations below.

$$
\begin{aligned}
\forall s \in S,\ \forall x \in X,\ \delta^z_{1,i}(t'_i)(s,x) &= \lambda_i(t'_i)[Id_S, z_{-i}]^{-1}(s,x) \\
&= \lambda_i(t'_i)[\{(s, l'_{-i}) : z_{-i}(l'_{-i}) = x_{-i}\}] \\
&= \lambda_i(\pi_i(t'_i))[\{(s, \pi_{-i}(l'_{-i})) : z_{-i}(l'_{-i}) = x_{-i}\}] \\
&= \lambda_i(t_i)[\{(s, l_{-i}) : z_{-i} \circ \pi^{-1}_{-i}(l_{-i}) = x_{-i}\}] \\
&= \delta^{z\circ\pi^{-1}}_{1,i}(t_i)(s,x).
\end{aligned}
$$

For the higher order belief, for all $i \in N$,

$$
\begin{aligned}
\forall E \in \Sigma(H^k_X),\ \delta^z_{k+1,i}(t'_i)[E] &= \lambda_i(t'_i)[Id_S, \delta^z_{k,-i}]^{-1}[E] \\
&= \lambda_i(t'_i)[\{(s, l'_{-i}) : (s, \delta^z_{k,-i}(l'_{-i})) \in E\}] \\
&= \lambda_i(\pi_i(t'_i))[\{(s, \pi_{-i}(l'_{-i})) : (s, \delta^z_{k,-i}(l'_{-i})) \in E\}]
\end{aligned}
$$

The induction hypothesis is that, for all $i \in N$, and any $t, t' \in T$ such that $t = \pi(t')$, $\delta^z_{k,i}(t'_i)(s,x) = \delta^{z\circ\pi^{-1}}_{k,i}(t_i)(s,x)$. Therefore,

$$
\begin{aligned}
\delta^z_{k+1,i}(t'_i)[E] &= \lambda_i(\pi_i(t'_i))[\{(s, \pi_{-i}(l'_{-i})) : (s, \delta^z_{k,-i}(l'_{-i})) \in E\}] \\
&= \lambda_i(t_i)[\{(s, l_{-i}) : (s, \delta^{z\circ\pi^{-1}}_{k,-i}(l_{-i})) \in E\}] \\
&= \delta^{z\circ\pi^{-1}}_{k,i}(t_i)(s,x).
\end{aligned}
$$

As a result, $\delta^z(t') = \delta^{z\circ\pi^{-i}}(t)$. Thus any image of t' induced by a signal z is always attained by its symmetric type t with a signal $z \circ \pi^{-1}$. $\square$

Proposition 4.14. *For $t, t' \in T$ in Λ, if $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$, then the smallest sub type spaces of Λ, Λ^t and $\Lambda^{t'}$, can be S-isomorphically embedded to the same space in $U(S \times C)$ where t and t' fall onto the same point.*

Proof. Suppose that $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$. Let Φ be the smallest sub type space of T which includes t, and let Φ' be the smallest sub type space of T which includes t'. We can pick the identity function on T for z. Then δ^z becomes the same as S-isomorphism to $U(S \times T)$, the universal type space on $S \times T$ in Liu[24]. Let $M(m)$ be the smallest sub type space on $U(S \times T)$. Then $M(\delta^z(t))$ is S-isomorphic to Φ. If $|\Phi| > |\Phi'|$, then $|\Phi'|$ can never mapped to $M(\delta^z(t))$ by any belief mapping. Therefore, we have $|\Phi| = |\Phi'|$.

By the assumption, there exists $z' : T \to T$ such that $z'(t') = \delta^z(t)$. Then we can consider Φ' as a Harsanyi type space on the payoff parameter $S \times T$. Since $|\Phi| = |\Phi'|$, $\delta^{z'}$ must be bijection from Φ' to $M(\delta^z(t))$. Therefore, Φ' does not have redundant types concerning $S \times T$. According to Mertens-Zamir, it implies that Φ' is $S \times T$-isomorphic to $M(\delta^z(t))$. Let $\lambda^{z'} \in \Delta(S \times T \times \Phi)$ which induced by λ and z', and $\mu_i \in \Delta(S \times T \times M_{-i}(\delta^z(t))$ be a natural belief mapping induced by Kolmogorov extension. Then,

$$\forall i \in N,\ \forall l_i \in \Phi_i,$$
$$\lambda^{z'}(l_i) = \mu_i(\delta_i^{z'}(l_i)) \circ [Id_{S \times T}, \delta_{-i}^{z'}]^{-1}.$$

Therefore,

$$\forall i \in N,\ \forall l_i \in \Phi_i,$$
$$Marg_(S \times \Phi')\lambda^{z'}(l_i) = Marg_(S \times M_{-i}(\delta^z(t)))\mu_i(\delta_i^{z'}(l_i)) \circ [Id_S, \delta_{-i}^{z'}]^{-1}.$$

By construction, $Marg_(S \times \Phi')\lambda_i^{z'} = \lambda_i$. Therefore, Φ' is S-isomorphic to $M_{-i}(\delta^z(t))$. Therefore, Φ' is S-isomorphic to Φ, and t' and t can be mapped to the same point S-isomorphically. $\square$

Under a plausible condition, we obtain the next result.

Lemma 4.15. *Let $t, t' \in T$ in Λ, and Λ is the smallest sub type space that includes t. Then, if $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$, t and t' are symmetric.*

Proof. By the above proposition, the smallest type space $\Lambda^{t'}$ is S-isomorphic to Λ. Therefore, by

Proposition 4.6., they are symmetric. □

We get the next theorem as a corollary.

Theorem 4.16. *Let $t, t' \in T$ in Λ, and Λ is the smallest sub type space includes t. Then, $BE(t, \Gamma) = BE(t', \Gamma)$ for any Γ if and only if they are symmetric to each other.*

4.3 Semantic interpretation of syntactic characterization

Sadzik adopted a first order language which can describe the modal logic of epistemology. He showed that $\{\delta^z(t) : z \in Z\} = \{\delta^z(t') : z \in Z\}$ if and only if the set of sentences which can be true by appropriate values of signals at the types are the same. The results in this section show that this syntactic characterization of types are equivalent to symmetry of types, and whether or not they can be mapped to the same sequential beliefs on $U(S \times C)$.

5 Application to intrinsic correlation

We have shown that any Harsanyi type space can be mapped isomorphically to a sub-space of $U(S \times C)$. One application of this theorem is the *intrinsic correlation* of beliefs proposed by Brandenburger-Friedenberg [10]. They showed that, in some complete information games, we cannot achieve all correlated rationalizable actions without any external mediator. They also showed that we can achieve all correlated rationalizable actions as intrinsic ones by adding a coin-flip to the basic uncertainty. In fact, their results are closely related to redundant types. In this section, we show the results of Brandenburger-Friedenberg in a different way; using redundant types and our theorems above.

5.1 Bayesian representation of correlated equilibrium

Consider a complete information game. Let $G \equiv \langle (A_i)_{i \in N}, (\pi_i)_{i \in N} \rangle$ be a game, where A_i is the strategy space of the agent i and $\pi_i : A \to \mathbb{R}_+$ is a payoff function. We assume that, for all $i \in N$, A_i is finite.[12] To define the correlated equilibrium of the game G, we introduce the Bayesian framework *a*

[12] This is the same assumption as Brandenburger-Friedenberg.

la Aumann. Let the basic uncertainty space be Ω, the information partition of the agent be $\mathscr{H}_i$, and the interim belief systems be $P(.|\mathscr{H}_i) \in \Delta(\Omega)$. Since A is finite, Ω can be chosen to be finite in order to represent correlated equilibria. [13]

Definition (Aumann [2]): For all $i \in N$, Let $f_i : \Omega \to A_i$ be measurable with regard to $\mathscr{H}_i$. Then $f \equiv (f_i)_{i \in N}$ is an *a posteriori equilibrium* iff

$$\forall i \in N, \forall \omega \in \Omega, \forall a_i \in A_i,$$
$$\sum_{\omega \in \Omega} \pi_i(f_i(\omega), f_{-i}(\omega)) \cdot P(\omega|\mathscr{H}_i(\omega)) \geq \sum_{\omega \in \Omega} \pi_i(a_i, f_{-i}(\omega)) \cdot P(\omega|\mathscr{H}_i(\omega)).$$

Definition (Bernheim [7], Pearce [27]): A set of strategies $R^\infty \subset \Pi_{i \in N} A_i$ is the set of the *correlated rationalizable actions* if (1)for each $i \in N$ and each $a_i \in R_i^\infty$, there exists $\mu \in \Delta(R_{-i}^\infty)$ such that a_i is a best response to μ, and (2) there is no set $F \subset \Pi_{i \in N} A_i$ such that it satisfies (1) and $F \supsetneq R^\infty$.

Concerning a posteriori equilibria and correlated rationalizable actions, we have the next equivalence result.

Proposition 5.1. *(Epstein [19] [14]) For any $a^* \in R^\infty$, there exists a posteriori equilibrium $\langle A, (\mathscr{H}_i)_{i \in N}, (P(.|\mathscr{H}_i))_{i \in N}, f \rangle$ such that, for all $i \in N$, $\mathscr{H}_i = A_i$, for all $a \in A$, $f(a) = a$, and $f(a^*) = a^*$.*

From $\langle A, (\mathscr{H}_i)_{i \in N}, (P(.|\mathscr{H}_i))_{i \in N}, f \rangle$, where $\mathscr{H}_i = A_i$, we can construct a Harsanyi type space on A. For all $i \in N$, let $T_i \equiv \mathscr{H}_i$. and $\lambda_i : T_i \to \Delta(A \times T)$ be as follows;

$$\begin{aligned} \lambda_i(a_i)[(a_{-i}, a_{-i})] = & P(a_{-i}|a_i) && \text{if } a_{-i} = f_{-i}(a_{-i}) \\ = & 0 && \textit{otherwise}. \end{aligned}$$

[13] As in the following argument, we can set Ω to be A.

[14] Aumann [3] and Brandenburger-Dekel [8] showed the same result.

Let $\Lambda \equiv \langle A, T, \lambda \rangle$. We can easily confirm that Λ is a Harsanyi type space on A.

Let $G' \equiv \langle \pi, \Lambda \rangle$ be a Bayesian game. For $i \in N$, let a strategy $\beta_i : T_i \to A_i$ be such that $\beta_i(t_i) = f_i(\omega)$ where $\omega \in t_i$. Then $\beta \equiv (\beta)_{i \in N}$ becomes a Bayesian Nash equilibrium of the game G'. The a posteriori equilibrium of the original game G is a Bayesian Nash equilibrium of G'.

5.2 Conditional independence and rationality and common certainty of rationality

Brandenburger-Friedenberg characterized intrinsic correlation by two conditions on Harsanyi types.

Definition: A Harsanyi type $t_i \in T_i$ satisfies *conditional independence* if $\lambda_i(t_i)[a_{-i}|h(t_{-i})] = \Pi_{j \in N \setminus \{i\}} \lambda_i(t_i)[a_j|h(t_{-i})]$, where h is the hierarchy mapping from $T \to U(A)$.

For the definition of another condition, *rationality and common certainty of rationality*, we need some preliminary definitions.

Definition: For each $i \in N$, a pair $(a_i, t_i) \in A_i \times T_i$ satisfies *rationality* if a_i is a best response to $Marg_{(A_{-i})} \lambda_i(t_i)$.

We use R_i to denote the set of the pairs that satisfies rationality.

Definition: For any $E \subset A_{-i} \times T_{-i}$, $t_i \in K_i(E)$ if $\lambda_i(t_i)[E] = 1$.

Definition: For each $i \in N$, $t_i \in T_i$ satisfies *rationality and common certainty of rationality* if $t_i \in R_i \cap \bigcap_{k=1}^{\infty} K^k(R)$, where K^k is the kth iteration of the operator K.

Since β is a BNE, it is almost clear that, for all $t_i \in T_i$, $(t_i, \beta_i(t_i))_{i \in N}$ satisfies RCBR. Polak [28] showed that RCBR is not sufficient condition for Nash equilibrium as shown in Aumann-Brandenburger [4], but Nash equilibrium satisfies RCBR under complete information about the payoffs. And he showed that the same thing applies to BNE. Here is a brief sketch of the proof. By construction, it is clear that, for all $i \in N$, $(t_i, \beta_i(t_i)) \in R_i^1$. Suppose that, for all $i \in N$ and $t_i \in T_i$, $(t_i, \beta_i(t_i)) \in R_i^k$. Then, since $\lambda_i(t_i)[\{(t_{-i}, a_{-i}) : a_{-i} = \beta_{-i}(t_{-i})\}] = 1$ and $\{(t_{-i}, a_{-i}) : a_{-i} = \beta_{-i}(t_{-i})\} \subset R_i^k$, we have $t_i \in B(R_{-i}^k)$. By the induction hypothesis, $(t_i, \beta_i(t_i)) \in R_i^k \cap [A_i \times B(R_{-i}^k)]$. Thus, $(t_i, \beta_i(t_i)) \in R_i^\infty$.

5.3 Conditional independence and redundancy

Note that conditional independence defined above is conditional on the sequential beliefs of the other agents' types. Therefore, when there are redundant types in Λ, it is hard for CI to be satisfied. However, the results that we have shown allows us to get rid of redundant types without affecting resulting equilibria.

In this section, we show first that, for any $a^* \in R^\infty$, there exists a Harsanyi type space Φ such that a^* is a realization of a BNE, and Φ has no *purely redundant types* except for one agent. As a result, we get the result that, for any $a^* \in R^\infty$, there exists a Bayesian formulation where a^* satisfies RCBR at a type which satisfies CI.

Proposition 5.2. *For any $a^* \in R^\infty$, there exists a posterior equilibrium such that, for some $\omega \in \Omega$, $f(\omega) = a^*$, and, for all $i \neq 1$, if $H_i \neq H_i'$, $P([a_j]_{j \neq i} | H_i) \neq P([a_j]_{j \neq i} | H_i')$ for some a_{-i}.*

Proof. By the proposition above, there exists a posterior equilibrium such that $\Omega = A$, for all $i \in N$, $\mathscr{H}_i = \{a_i \times A_{-i} : a_i \in A_i\}$ and $f_i(a) = a_i$. Let this a posteriori equilibrium be $\mathcal{F}$ and $[a_i] \equiv a_i \times A_{-i}$. For notational convenience, we denote each class in the agent i's information partition as $[a_i]$. Now it is possible that there exists $[a_i] \neq [a_i']$ such that, for all H_{-i}, $P(H_{-i} | [a_i]) = P(H_{-i} | [a_i'])$. Then we can duplicate the agent 1's information partition.

Suppose that, for $b_1 \in A_1$, $P([b_1] | [a_i]) > 0$. We add another set of states so that the states of the world $\hat{\Omega} = (A_1 \cup \{a_1^2\}) \times A_{-1}$ and associate another information partition $[a_1^2]$ to the agent 1. We define a new a posterior equilibrium $\hat{\mathscr{F}} \equiv \langle A, \hat{\Omega}, \hat{P}, \hat{f} \rangle$ such that

$$For\ j = 1, \qquad \hat{f}_1([a_1^2]) = b_1$$

$$\forall a_1 \in A_1,\ \hat{f}_1([a_1]) = f_1([a_1]).$$

$$\forall j \neq 1, \forall a_j \in A_j,\ \hat{f}_j([a_j]) = f_j([a_j]).$$

The new sequence of conditional beliefs is defined in the following way;

$$For\ j = 1, \qquad \hat{P}(.|[a_1^2]) = P(.|[a_1])$$

$$\hat{P}(.|[a_1]) = P(.|[a_1])\ otherwise.$$

$$\forall\ j \neq 1, \forall a_j \neq [a_i'], \qquad \hat{P}(.|[a_j]) = P(.|[a_j])$$

$$For\ j = i\ and\ a_i',\ \forall a_{-1,i} \in A_{-1,i}, \qquad \hat{P}((a_1^2, a_{-1,i})|[a_i']) = P((b_1, a_{-1,i})|[a_i'])$$

$$\hat{P}(.|[a_i']) = P(.|[a_i'])\ otherwise.$$

It is easy to show that $\hat{\mathscr{F}} \equiv \langle A, \hat{\Omega}, \hat{P}, \hat{f} \rangle$ is an a posteriori equilibrium, and $\hat{f}(a^*) = a^*$. Note that, in this a posterior equilibrium, $\hat{P}(.|[a_i])$ and $\hat{P}(.|[a_i'])$ are distinguishable at the event $[b_i]$.

Since N and A are finite, we can iterating this process until every pair of each agent's, except for the agent 1, information states $[a_j] \neq [a_j']$ have different conditional beliefs over the other players' information states. $\square$

Corollary 5.3. *For any $a^* \in R^\infty$, there exists a Harsanyi type space $\Lambda = \langle A, T, \lambda \rangle$ and a pair of Bayesian equilibrium strategy $\beta = (\beta_i)_{i \in N}$ such that $a^* = \beta(t)$, and, for all $i \neq 1$, T_i has no purely redundant types.*

Then we can apply the theorem to find an S-isomorphic Harsanyi type space Φ on $A \times \{0, 1\}$ which has no redundant types. And, in Φ, no types result in the same sequential belief. Therefore, each type and its action associated by the equilibrium strategy β satisfy CI. Therefore we have the next result, which is the same result shown in a different way by Brandenburger-Friedenberg.

Theorem 5.4. *For any $a^* \in R^\infty$, there exists a Harsanyi type space $\Phi = \langle A \times \{0, 1\}, V, \phi \rangle$ such that a^* satisfies RCBR at some state $v \in V$ which satisfies CI.*

6 Conclusion

In this paper, we showed that it is possible to embed Harsanyi type spaces isomorphically onto the space of sequential beliefs over an augmented uncertainty, even if they have redundant types. The technique to introduce a payoff irrelevant parameter is an extension of Liu. However we have the following distinctions: (1) our payoff irrelevant parameter space is exogenous, and (2) it is enough that the parameter space has only two values. That is, any correlation of types in Bayesian frameworks which cannot be explained by the basic uncertainty is resolved by adding a coin flip to the uncertainty. Concerning the first finding, the exogeneity of the parameter allowed us to show the existence of the universal type space where the vast majority of Harsanyi type spaces are uniquely embedded.

We showed that our results can be applied to provide a characterization of Bayesian Equilibrium and an interpretation of intrinsic correlation in games. Because we presented a universal type space that includes reduntant type spaces, the recent research on strategic topologies on Mertens-Zamir's universal type space can potentially be extended to our space. Moreover, we can use Bayesian Equilibrium as the solution concept, differently from Dekel et al who used ICR and Ely-Peski who used IIR.

7 Appendix

7.1 Bimeasurability of the function ϕ

Let $\Phi = \langle S \times C, V_1 \times V_2, (\phi_i)_{i\in\{1,2\}}\rangle$ be a Harsanyi type space, $V_1 = V_2 = [0,1]$, and $C = \{0,1\}$ as defined in the section 3. First we show that $\phi_1 : V_1 \to \Delta(S \times V_2 \times C)$ is bimeasurable. It is worth while to notice that ϕ_1 maps each element in V_1 to a product measure on the measurable space $(\ S \times V_2 \times C,\ \Sigma(S \times V_2 \times C)\ )$.

We define the following functions.

$$f_1 : V_1 \to \Delta(S \times V_2) \text{ such that } f_1(v_1) = \lambda_1(p_1^{-1}) \circ [Id_S, p_2]^{-1}.$$

$$g_1 : V_1 \to \Delta(C) \text{ such that } g_1(v_1)(0) = v_1.$$

You can see that both f_1 and g_1 are bimeasurable functions. Then we have that $\phi_1(v_1) = f_1(v_1)\times g_1(v_1)$, where $f_1(v_1)\times g_1(v_1)$ is the product measure on the Borel measure space $(\ S\times V_2\times C,\ \Sigma(S\times V_2\times C)\ )$.[15] Since $S \times V_2$ and C are both second countable, $\Sigma(S \times V_2) \times \Sigma(C) = \Sigma(S \times V_2 \times C)$.

Lemma 7.1. *The Borel σ-algebra* $\Sigma(S \times V_2 \times C) = \{E : \exists A \in \Sigma(S \times V_2),\ \exists B \in \Sigma(C),\ E = A \times B\}$.

Proof. Let $\hat{\Sigma} \equiv \{E : \exists A \in \Sigma(S \times V_2),\ \exists B \in \Sigma(C),\ E = A \times B\}$. We only have to show that $\hat{\Sigma}$ is a σ-algebra. It is clear that $\varnothing,\ S \times V_2 \times C \in \hat{\Sigma}$. Let $E \in \hat{\Sigma}$. Then there exists $A \in \Sigma(S \times V_2)$ and $B \in \Sigma(C)$ such that $E = A \times B$. Therefore $E^c = A^c \times C \cup A \times B^c$.

Let $\Delta^P(S \times V_2 \times C) \subset \Delta(S \times V_2 \times C)$ be the set of the product measures over $S \times V_2$ and C.

Lemma 7.2. *The subspace $\Delta^P(S \times V_2 \times C)$ is homeomorphic to the product space $\Delta(S \times V_2) \times \Delta(C)$.*

[15] By Caratheodory's extension theorem, the product measure is uniquely determined.

Proof. By Caratheodory's extension theorem, the function $d : \Delta(S \times V_2) \times \Delta(C) \to \Delta^P(S \times V_2 \times C)$ such that $(\eta, \mu) \mapsto \eta \times \mu$ is bijection.

First we want to show that d is a continuous function. The topological base of $S \times V_2 \times C$ is $t = \{G \times a : G \text{ is an open subset of } S \times V_2, \text{and } a \in 2^C\}$. Therefore any open set $G' \subset S \times V_2 \times C$ takes the form of

$$G' = \tilde{G}_1 \times \{0\} \cup \tilde{G}_2 \times \{1\} \cup \tilde{G}_3 \times \{0, 1\},$$

where, for $i = 1, 2, 3,\ \ \tilde{G}_i$ is an open set in $S \times V_2$. It is reduced to

$$G' = G_1 \times \{0\} \cup G_2 \times \{1\},$$

where, for $i = 1, 2,\ \ G_i$ is an open set in $S \times V_2$.

Let $\{\eta_\alpha\}$ be a net in $\Delta(S \times V_2)$ such that $\eta_\alpha \to \eta$. And let $\{\mu_\alpha\}$ be a net in $\Delta(C)$ such that $\mu_\alpha \to \mu$. Then,

$$\begin{aligned} &\forall G: \text{ open}, \ \mathit{lim\ inf}\ \eta_\alpha(G) \geq \eta(G), \\ &\forall a \in 2^C, \ \mathit{lim\ inf}\ \mu_\alpha(a) \geq \mu(a). \end{aligned}$$

Let $\nu_\alpha \equiv \eta_\alpha \times \mu_\alpha$, and $\nu = \eta \times \mu$. Then, for each open set $G' \subset S \times V_2 \times C$,

$$\begin{aligned} \nu_\alpha(G') &= \nu_\alpha(G_1 \times \{0\}) + \nu_\alpha(G_2 \times \{1\}) \\ &= \eta_\alpha(G_1)\mu_\alpha(\{0\}) + \eta_\alpha(G_2)\mu_\alpha(\{1\}). \end{aligned}$$

In the same way,

$$\nu(G') = \eta(G_1)\mu(\{0\}) + \eta(G_2)\mu(\{1\}).$$

Since $\eta_\alpha \to \eta$ and $\mu_\alpha \to \mu$,

$$lim\ inf\ \eta_\alpha(G_1)\mu_\alpha(\{0\}) \geq \eta(G_1)\mu(\{0\}).$$
$$lim\ inf\ \eta_\alpha(G_2)\mu_\alpha(\{1\}) \geq \eta(G_2)\mu(\{1\}).$$

And,

$$\begin{aligned} lim\ inf\ \nu_\alpha(G') &= lim\ inf\{\eta_\alpha(G_1)\mu_\alpha(\{0\}) + \eta_\alpha(G_2)\mu_\alpha(\{1\})\} \\ &\geq lim\ inf\ \eta_\alpha(G_1)\mu_\alpha(\{0\}) + lim\ inf\ \eta_\alpha(G_2)\mu_\alpha(\{1\}) \\ &\geq \eta(G_1)\mu(\{0\}) + \eta(G_2)\mu(\{1\}) \\ &= \nu(G'). \end{aligned}$$

Therefore, $\nu_\alpha \to \nu$. Therefore d is a continuous function.

Next, we show that d^{-1} is a continuous function. Let $\{\nu_\alpha\} \equiv \{\eta_\alpha \times \mu_\alpha\}$ be a net of product measures such that $\nu_\alpha \to \nu = \eta \times \mu$. Then, $\nu_\alpha(S \times V_2 \times a) = \mu_\alpha(a)$, and $\nu(S \times V_2 \times a) = \mu(a)$. Since $lim\ inf\ \nu_\alpha(S \times V_2 \times a) \geq \nu(S \times V_2 \times a)$, $lim\ inf\ \mu_\alpha(a) \geq \mu(a)$. In the symmetric way, $lim\ inf\ \eta_\alpha(G) \geq \eta(G)$. It means that $(\eta_\alpha,\ \mu_\alpha) \to (\eta,\ \mu)$. Therefore d^{-1} is a continuous function. $\square$

Corollary 7.3. *The subspace $\Delta^P(S \times V_2 \times C)$ is closed.*

Since $\Delta(S \times V_2) \times \Delta(C)$ is second countable, $\Sigma(\Delta(S \times V_2) \times \Delta(C)) = \Sigma(\Delta(S \times V_2)) \times \Sigma(\Delta(C))$

Proposition 7.4. *The function $\phi_1 : V_1 \to \Delta(S \times V_2) \times \Delta(C)$ is a bimeasurable function.*

Proof. (*Inverse measurability*) Let $\phi_1 = (f_1, g_1)$. The space of the probability measures $\Delta(C)$ is homeomorphic to V_1, and g_1 is its homeomorphism. We consider that $\phi_1 : V_1 \to \Delta(S \times V_2) \times V_1$ and $\phi_1 = (f_1, Id)$. It allows us to consider that $\phi_1(V_1) \subset \Delta(S \times V_2) \times V_1$ is a graph of the function f_1^{-1}. Since f_1^{-1} is a measurable function, the graph $\phi_1(V_1)$ is a Borel set in the product measure space $\Delta(S \times V_2) \times V_1$.[16] For each $E \in \Sigma(V_1)$, $\phi_1(E) = f_1(E) \times E \cap \phi(V_1)$. We know that both $f_1(E) \times E$ and $\phi(V_1)$ are measurable. Therefore, $\phi_1(E)$ is measurable.

(*Measurablity*) Let $E \subset \Delta(S \times V_2) \times V_1$ be a rectangle. Let π_1 and π_2 be the projection onto V_1 and $\Delta(S \times V_2)$ respectively. Let $F_2 \equiv f_1 \circ \pi_1(E) \subset \Delta(S \times V_2)$. Since f_1 is bimeasurable, F_2 is also Borel. For each $y \in \pi_2(E)$, $f_1^{-1}(y) \in \pi_1(E)$ if and only if $y \in \pi_2(E) \cap F_2$. Let Therefore, the intersection of the rectangle E and the entire graph $\phi_1(V_1) \equiv \{(x, f_1(x)) : x \in V_1\}$ becomes $G \equiv \{(f_1^{-1}(y), y) : y \in \pi_2(E) \cap F_2\}$. Since $\pi_2(E)$ and F_2 are both Borel, $\pi_2(E) \cap F_2$ is also Borel. We can see that $\phi_1^{-1}(E) = \pi_1(G)$. Since f_1 is measurable, $\pi_1(G)$ is also Borel. Therefore $\phi_1^{-1}(E)$ is Borel. □

Proposition 7.5. *The function $\phi_2 : V_2 \to \Delta(S \times V_1 \times \Delta(C))$ is a bimeasurable function.*

Proof. Let

$$\Delta_0 \equiv \{\mu \in \Delta(S \times V_1 \times C) : \forall E \in \Sigma(S \times V_1),\ \mu(E \times \{1\}) = 0\}.$$

[16] See Halmos [21] pp143.

Let $f : S \times V_1 \times C \to \mathbb{R}$ such that

$$\forall e \in S \times V_1,\ f(e, 0) = a,$$
$$f(e, 1) = b.$$

Then, $f \in C_b(S \times V_1 \times C)$. Therefore, when a net $\{\mu_\alpha\}$ converges to some probability measure, it must be in Δ_0. Therefore, Δ_0 is a closed set. Since λ_2 is bimeasurable between V_2 and $\Delta(S \times V_1)$ and $\Delta(S \times V_1)$ is homeomorphic to Δ_0, ϕ_2 is bimeasurable between V_2 and Δ_0. $\square$

I use the term "bimeasurable" in a slightly different way.

Definition: A function $f : X \to Y$ is bimeasurable if f is measurable and , for each measurable set $E \subset X$, $f(E)$ is also measurable.

Lemma 7.6. *Let X and Y be Polish, and $f_1 : X \to \Delta(X)$ and $g_1 : X \to Y$ be both bimeasurable. Let $f_2 : X \to \Delta(Y)$ be such that, for each $x \in X$, $f_2(x) = f_1(x) \circ g_1^{-1}$. Then, the function f_2 is bimeasurable.*

Proof. The measurability of f_2 is shown by Liu.[17] We only show that, for all $E \in \Sigma(X)$, $f_2(E) \in \Sigma(\Delta(Y))$.

Let $g_2 : \Delta(X) \to \Delta(Y)$ such that, for all $\mu \in \Delta(X)$, $g_2 : \mu \mapsto \mu \circ g_1^{-1}$. Let $A \equiv \{\mu \in \Delta(X) : \mu(E) \geq p\}$, where $E \in \Sigma(X)$ and $p \in [0, 1]$. Then, $g_2(A) = \{\nu \in \Delta(Y) : v(g_1(E)) \geq p\} \cap \{\nu \in \Delta(Y) : \nu(g_1(X)) = 1\}$. Notice that $g_1(X) \in \Sigma(Y)$. The both sets in the right hand side of the equation are measurable.

[17] See Lemma 5 in Liu [24]

Therefore $g_2(A) \in \Sigma(Y)$.

Since $f_2 = g_2 \circ f_1$, we have that, for each $E \in \Sigma(X)$, $f_2(E) \in \Sigma(\Delta(Y))$. $\square$

Lemma 7.7. *For each $k \geq 1$, the kth order hierarchy mapping h_i^k is bimeasurable.*

Proof. Without loss of generality, we only have to show that h_1^k is bimeasurable.

For $k = 1$, let $X \equiv S \times V \times C$, $Y \equiv S \times C$, $f_1 \equiv \tilde{\phi}_i$, and $g_1 \equiv proj_{(S \times C)}$, where, for all $(s, c, v_1, v_2) \in S \times V \times C$, $\tilde{\phi}_i(s, c, v_1, v_2) \equiv \phi_1(v_1)$. It is easy to see that f_1 and g_1 are bimeasurable. By the lemma, $\tilde{h}_1^1 : S \times V \times C \to \Delta(S \times C)$ is bimeasurable. We can just restrict the domain from $S \times V \times C$ to V_1 to get the h_1^1 which is measurable.

For $k \geq 2$, we assume that, for $i = 1, 2$, h_i^{k-1} is bimeasurable as the induction hypothesis. Let $X \equiv S \times V \times C$, $Y \equiv S \times C \times H^k(S \times C)$, $f_1 \equiv \tilde{\phi}_1$, and $g_1 \equiv \tilde{h}_2^{k-1}$, where $\tilde{h}_2^{k-1}(s, c, v_1, v_2) \equiv h_2^{k-1}(v_2)$. By the lemma, $\tilde{h}_1^k : S \times V \times C \to \Delta(S \times C)$ is bimeasurable. We can just restrict the domain from $S \times V \times C$ to V_1 to get the h_1^k which is measurable. $\square$

Proposition 7.8. *The full hierarchy mapping h_i is bimeasurable.*

Proof. First, we show that h_i is measurable. The σ-algebra on $\Pi_{k=1}^{\infty} H^k$ is the σ-algebra generated by

$$\mathcal{F} \equiv \{F = \Pi_{k \notin I} E_k \times \Pi_{k \in I} H^k : I \subset \mathbb{N} \text{ is finite, and } E_k \in \Sigma(H^k)\}.$$

Since $h_i \equiv (h_i^1, \ \ldots, \ )$, for each $F \in \mathcal{F}$, $h_i^{-1}(F) \in \Sigma(V_i)$. By Theorem 4-1-6 in Dudley [16], h_i is measurable.

Next, we show that, for each $E \in \Sigma(V_i)$, $h_i(E)$ is measurable. Let $E \in \Sigma(V_i)$. Since h_i^1 and h_i^2 are bimeasurable injection, $h_i^2 \circ (h_i^1)^{-1}$ is bimeasurable bijection from $h_i^1(E)$ to $h_i^2(E)$. Let the image of E by (h_i^1, h_i^2) be $\Gamma_2(E)$. It means $\Gamma_2(E) \equiv \{(h_i^1(v_i), h_i^2(v_i)) \in H^1 \times H^2 : v_i \in E\}$. We can see that it is the graph of $h_i^2 \circ (h_i^1)^{-1}$. Therefore, $\Gamma_2(E)$ is measurable in the product measurable space $H^1 \times H^2$. By the mathematical induction, for each $k \geq 1$, $\Gamma_k(E) \subset \Pi_{l=1}^k H^l$, the image of E by $(h_i^1, \cdots, h_i^k)$, is measurable. The image of the full hierarchy $h_i(E)$ is the projective limit of $(\Gamma_k(E))_{k\in\mathbb{N}}$, and as we saw, each $\Gamma_k(E)$ is measurable. Therefore $h_i(E)$ is measurable. □

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