On the Equivalence of Iterated Application of a Choice Rule and Common Belief of Applying that Rule

Michael Trost*

Albert-Ludwigs-Universität Freiburg

Working Paper†

Version: April 15, 2010

Abstract

In this paper solution concepts originating from an iterated application of some choice rule (IACR) are analyzed from a decision-theoretic perspective. The question is whether the solutions generated by IACR coincide with the solutions resulting from choice rule following behavior and common belief of that. In general, this equivalence does not hold, but we specify conditions on choice rules which ensure it. The conditions are positive, internal, conditional and marginal consistency. Prominent examples of choice rules satisfying those conditions are strict undominance in pure acts, strict undominance in mixed acts and Börgers' undominance concept. Furthermore, by providing examples, it is established that our result is weak in the sense that none of those conditions can be canceled without breaking up the above epistemic characterization of IACR.

JEL Classification Number: C72, D81.

Key words: Choice rule, common belief, epistemic game theory, decision theory

---

*Address: Institute for Research in Economic Evolution, Alberts-Ludwig-Universität Freiburg, Platz der Alten Synagoge, KG II, 79085 Freiburg im Breisgau, Germany. E-Mail: michael.trost@vwl.uni-freiburg.de.

†This paper has been submitted to the Workshop on Epistemic Game Theory of the 21st Stony Brook Game Theory Festival on July 19 - 20, 2010. Updated versions of this paper will be available on the author's website `http://portal.uni-freiburg.de/erwien/staff/doctorial-candidates-1/trost`. Any comments are gratefully received.

1 Introduction

Solution concepts for strategic games which originate from an iterated application of some choice rule (IACR) are well-known in game theory and often applied to solving situations of interaction. Prominent examples are the iterated elimination of strictly dominated strategies, iterated elimination of weakly dominated strategies, the (uncorrelated) rationalizability concept of Bernheim (1984) and Pearce (1984) and the iterated application of the pure undominance concept of Börgers (1993). Recently, Halpern and Pass (2009) propose the iterated application of the minimax regret rule. Latter choice rule has been put forward by Niehans (1948) and Savage (1951) and singles out those alternatives which minimize the maximal deviation from the maximal attainable payoff.

In this paper we analyze those solution concepts from a decision-theoretic point of view. We ask whether such concepts can be characterized by choice rule following behavior of the players and common belief among them that every player follows that choice rule. Or more precisely, the question is whether the outcomes obtained by IACR coincide with the outcomes occurring when every player applies this choice rule on her strategic decision problem and there is common belief among them that they apply it. Note, such equivalence comprises two requirements. For the one thing, it is required that any strategy profile surviving IACR has to be a possible outcome under choice rule following behavior and common belief of that. For the other thing, any outcome resulting from choice rule following behavior and common belief of that has to survive IACR. As we will demonstrate below both requirements are generally not satisfied. This negative result is the starting point of this paper. Our objective is to specify conditions on the choice rules entailing that this equivalence holds. Clearly, we are interested that these conditions are as weak as possible. By providing examples it is established that our results are weak in the sense that none of our conditions can be canceled without breaking down this equivalence.

In order to understand that this equivalence is by no means guaranteed consider the strategic game depicted in figure 1. The game is solved by iterated application of the maximin rule, where at each round *all* strategies are deleted which are unfavorable according to that rule. It turns out that at the first round the strategies u and m of player R are deleted and at the second round strategy r of player C is deleted. Then the deletion process stops. Only strategy profile (d, l) survives the iterated application of the maximin rule.

As mentioned above to check equivalence between IACR and the decision theoretic assumptions of choice rule following behavior and common belief of that we have to address two issues with regard to this solution. Firstly, we ask whether this outcome can result from maximin rule following behavior and common belief of that (as usual, latter means that every player believes that every player follows the maximin rule, that every

		Player C	
		l	r
	u	$(0, 0)$	$(0, 0)$
Player R	m	$(4, 2)$	$(1, 1)$
	d	$(2, 2)$	$(3, 1)$

Figure 1: Strategic game Γ_1

player believes that every player believes that every player follows the maximin rule, and so on). Secondly, we examine whether strategy profiles exist which do not survive the iterated application of the maximin rule, but are consistent with these assumptions about the players' behavior and beliefs.

Let us start with the first issue and suppose that the above solution would be realized (i.e. R plays d and C plays l), that both players would act according to the maximin rule and that there would be common belief among them that both apply that rule. Then R's choice for playing d can only result if she considers as possible that her opponent C chooses r (otherwise she would choose strategy m). Note it is supposed that R believes that C applies the maximin rule. Therefore R believes that C considers as possible that R chooses u. Common belief of applying the maximin rule also implies that R believes that C believes that R applies the maximin rule. However, regardless what strategies of player C player R considers as possible, R never chooses u according to the maximin rule. For this reason common belief of applying the maximin rule is inconsistent with a belief of R that C considers possible that R chooses u. Hence, the strategy profile (d, l) obtained by iterated application of the maximin rule can never be realized if every player applies this rule and there is common belief of that.

The preceding discussion has shown that not any strategy profile surviving the iterated application of the maximin rule is consistent with maximin rule following behavior and common belief of that. Let us turn to the second issue put forward above and examine the other direction. We ask whether any strategy profile resulting from maximin rule following behavior and common belief of that survives the iterated application of that rule. Return to the above strategic game Γ_1 and suppose that R plays strategy m and C plays strategy l. Furthermore, suppose that both players considers as possible only the actual state (i.e. the state at which they play m and l, respectively). Then at this state (*i*) both players believe that R plays m and C strategy l, (*ii*) both players believe (*i*), (*iii*) both players believe (*ii*), and so on. Statement (*i*) implies that their actual choices m and l are favorable according to the maximin rule. Statement (*ii*) implies that at the actual state they believe that their choices are favorable according to the maximin rule, and statement (*iii*) implies that at the actual state they believe that they believe that their choices are favorable according to the maximin rule, and so on. In other words, at the actual state the strategy profile (m, r) is realized, both players act according to the maximin rule and there is common belief among them that they act according to the maximin rule. Although strategy profile (m, r) does not survive the iterated application of the maximin rule on Γ_1, it can be realized if both player apply the maximin rule and there is common belief of that.

As just exemplified the outcome generated by the iterated application of some choice rule can differ from that realized by players acting according to this rule and having common belief of that. The goal of this paper is to specify conditions on choice rules such that the two outcomes coincides. Thereby the solution concept of IACR gains an epistemic foundation.

We proceed as follows. In the succeeding section we introduce the solution concept of IACR and lists its basic properties. The objective of section 3 is to state conditions on choice rules such that the solution generated by IACR has the best choice property. This property is a generalization of the best response property introduced by Pearce (1984) to characterize his solution concept of rationalizability. In section 4 we supplement the strategic game with an epistemic model and examine under which conditions IACR and common belief of applying this choice rule coincides. Concluding remarks to our results are given in section 5.

2 Iterated Application of Choice Rules to Strategic Games

In this section we describe in detail the process of iterated application of some choice rule (IACR) on finite strategic games. At first we consider choice rules for any kind of subjective uncertainty. We follow the state space framework proposed by Savage (1954) and describe uncertainty by a set of possible states of the world. Each state represents a specific resolution of all uncertain features relevant to the decision maker. This framework is fundamental to our analysis. In the remaining part of this section we consider finite strategic games and decompose them into individual decision problems under uncertainty. Based on this decomposition the choice rule is iteratively applied on the strategic game.

Following Savage (1954) the uncertainty of an decision maker (e.g. the uncertainty originating from participating in a game) is represented by a state space Ω. It is a finite set consisting of all possible states of the world. Each state represents a specific resolution of all uncertain features relevant to the decision maker. A choice of a decision maker induces a specific profile of outcomes on the state space. Henceforth, we assume that the outcomes are real numbers and could be interpreted as monetary payoffs. Formally, a choice is a mapping assigning to each state of the world a real-valued payoff. The set of all these mappings is denoted by $\mathbb{R}^\Omega$. In literature these mappings are known as *acts*. The ω-th component of act x is denoted by x_ω and indicates the payoff the decision maker receives when she has chosen act x and state ω occurs. A subset E of Ω is termed as *event* and the restriction of some act $x \in \mathbb{R}^\Omega$ on E is denoted by $x|_E$. Note this notation differs slightly from the notation generally found in decision theory, where restriction on E is simply denoted by x_E. We deviate from this standard to save coherency to the following conventions.

A non-empty finite subset C of $\mathbb{R}^\Omega$ is termed *constraint* and shall comprise all acts that are available for the decision maker. A *decision problem under uncertainty* is described by a pair $\Phi := (\Omega, C)$, where Ω is the state space specifying the uncertainty the decision maker is faced with and C the constraint specifying the options the decision maker has at his disposal. Let $E \subseteq \Omega$ be a non-empty event and $\Phi := (\Omega, C)$ some decision problem under uncertainty, then $C|_E := \{x|_E : x \in C\}$ denotes the constraint reduced on E and $\Phi|_E := (E, C|_E)$ denotes the decision problem Φ reduced on event E. Latter can be understood as the new decision problem arising when the decision maker confide in the information that the actual state of the world belongs to E.

A *choice rule* $\mathcal{C}$ is a mapping that assigns to each decision problem $\Phi := (\Omega, C)$ a (possibly empty) set $\mathcal{C}(\Phi) \subseteq C$ of acts. The set $\mathcal{C}(\Phi)$ is called *choice set* and the acts belonging to this set are termed as *favorable* or *best acts* under choice rule $\mathcal{C}$. Available acts which do not belong to the choice set are called *unfavorable acts*.

Up to now it has been left open what is hidden behind a state of the world. Now we concretize the environment the decision maker is faced with. Henceforth, we consider situations at which a group of decision makers interact, that is, a situation at which the payoffs decision makers receive are affected not only by their own choice, but also by the choice of the other decision makers. Such situations are called games and the rules how this interaction takes place are recorded in the game form. In this paper we restrict ourselves on the most simple game form, the so-called strategic game. This class of games is characterized by the property that individuals decide only once and simultaneously (i.e. no individual has observed the decisions of the other individuals, when she decides to move).

Formally, a strategic game Γ is described by a tuple $\Gamma := (S^i, z^i)_{i\in N}$, where N denotes a non-empty, finite set of players, S^i a non-empty set of strategies for player i, and $z^i : \times_{j\in N} S^j \to \mathbb{R}$ player i's payoff function. We presuppose that the strategic game is finite, that is, S^i is finite for every player $i \in N$. A member s^i of S^i is referred to as strategy for player i and the combination $(s^i)_{i\in N}$ of players' strategies as strategy profile. Observe payoff function z^i assigns to every strategy profile $s := (s^i)_{i\in N}$ a real-valued number $z^i\left((s^i)_{i\in N}\right)$ which could be interpreted as monetary payoff. As usual, the set of all strategy profiles is denoted by $S := \times_{i\in N} S^i$ and the set of all profiles listing strategies of players different to i by $S^{-i} = \times_{j\in N\setminus\{i\}} S^j$. Consider a strategic game Γ and let $\tilde{S}^i \subseteq S^i$ for every player $i \in N$. The strategic game $\Gamma|_{\tilde{S}} := \left(\tilde{S}^i, z^i|_{\tilde{S}}\right)_{i\in N}$, where $z^i|_{\tilde{S}}$ denotes the restriction of the payoff function on the domain $\tilde{S} := \times_{i \in N} \tilde{S}^i$, is called the *reduction of game* Γ *on strategy space* $\tilde{S}$.

The rules of a game capture all exterior circumstances under which the interaction takes place. In traditional game theory, they constitute the basis upon which a game theorist build her prediction on the outcome of the game. This standard procedure is formally reflected in solution concepts. In general, a *solution concept* is a mapping that assigns to each game of some class of games a (possibly empty) set of strategy profiles which are called solutions of the game. The focus of traditional game theory is to figure out solution concepts which have desirable properties (like non-emptiness or uniqueness) or which fits in with empirical results. In this paper we obey to the traditional approach until section 4. Up to that point we base our predictions of the outcome of the game on solution concepts resulting from an iterated application of choice rules like those presented above. In section 4 we supplement the strategic game with a formal framework which describes explicitly the reasoning of the players about themselves and their opponents. There we will examine whether the solution generated by iterated application of some choice rule coincides with the solution under choice rule following behavior and common belief of that.

Implementing some choice rule on some strategic game Γ requires to decompose the game into individual decision problems. Thereto each strategy $s^i \in S^i$ is associated with the act z^{i,s^i} on S^{-i} determined by $z^{i,s^i}_{s^{-i}} := z^i(s_i, s_{-i})$ for all $s^{-i} \in S^{-i}$. This act also referred to as the *act induced by strategy* s^i gives the payoffs player i receives for each strategy combination of the other players when she selects strategy s^i. Note, we often identify such acts by the same letter as the strategy that induces this act. This should cause no problems, because it should be clear from the context whether the strategy or its induced act is meant. The set of available acts for player i participating in game Γ is given by $C^i_\Gamma := \{z^{i,s^i} \in \mathbb{R}^{S^{-i}} : s^i \in S^i\}$. Again we often slightly abuse the notation and denote this constraint also by the letter labeling player i's strategy set. Following this convention the decision problem of player i is described by the tuple $\Phi^i_\Gamma := (S^{-i}, S^i)$. This pair is referred to as the *strategic decision problem of player* i *given strategic game* Γ.

Suppose a choice rule signed by $\mathcal{C}$ is applied on those decision problems. In order to express their solutions in terms of strategies rather than acts we have to introduce the mapping $\gamma^i_\Gamma : S^i \to \mathbb{R}^{S^{-i}}$ which assigns to each strategy of player i available in game Γ its induced act, that is $\gamma^i_\Gamma(s^i) := z^{i,s^i}$ for all $s^i \in S^i$. Now, fix

$$\mathcal{R}^i_1(\Gamma) := (\gamma^i_\Gamma)^{-1}(\mathcal{C}(\Phi^i_\Gamma))$$

for any player $i \in N$. The set $\mathcal{R}^i_1(\Gamma)$ consists exactly of those strategies of player i which induce acts that are solutions of her decision problem according to choice rule $\mathcal{C}$. These sets are the starting point of the following elimination process triggered by choice rule $\mathcal{C}$.

Definition 2.1 *Consider a finite strategic game* Γ *and let* $\mathcal{C}$ *a choice rule for the corresponding individual strategic decision problems. The process of iterated application of choice rule* $\mathcal{C}$ *on* Γ *is a sequence* $(\mathcal{R}_k(\Gamma))_{k\in\mathbb{N}}$ *of sets of strategy profiles inductively determined by*

$$\mathcal{R}_1(\Gamma) := \times_{i\in N}\mathcal{R}_1^i(\Gamma)$$

and for all $k \geq 1$

$$\mathcal{R}_{k+1}(\Gamma) := \mathcal{R}_1(\Gamma|_{\mathcal{R}_k(\Gamma)}) \ .$$

We say to a strategy profile $s \in \mathcal{R}_k(\Gamma)$ *that it survives* k *rounds of elimination of* $\mathcal{C}$*-unfavorable strategy profiles. The set*

$$\mathcal{R}_\infty(\Gamma) := \cap_{k\in\mathbb{N}}\mathcal{R}_k(\Gamma)$$

is referred to as the set of strategy profiles that survive the iterated application of choice rule $\mathcal{C}$ *on* Γ.

Similarly, for each player i we inductively define the sequence $\left(\mathcal{R}_k^i(\Gamma)\right)_{k\in\mathbb{N}}$ of sets of strategies by $\mathcal{R}_{k+1}^i(\Gamma) := \mathcal{R}_1^i(\Gamma|_{\mathcal{R}_k(\Gamma)})$ for any $k \in \mathbb{N}$. Because

$$\mathcal{R}_{k+1}(\Gamma) = \mathcal{R}_1(\Gamma|_{\mathcal{R}_k(\Gamma)}) = \times_{i\in N}\mathcal{R}_1^i(\Gamma|_{\mathcal{R}_k(\Gamma)}) = \times_{i\in N}\mathcal{R}_{k+1}^i(\Gamma)$$

holds for any $k \in \mathbb{N}$, this sequence consists exactly of those sets of strategies of player i which constitute the i-th component of the elimination process for strategy profiles introduced in definition 2.1. Therefore it is justified to say to strategies belonging to $\mathcal{R}_k^i(\Gamma)$ that they survive k rounds of elimination of $\mathcal{C}$-unfavorable strategies. Observe, by definition, $l \geq k$ implies $\mathcal{R}_k(\Gamma) \supseteq \mathcal{R}_l(\Gamma)$ and, hence, $\mathcal{R}_k^i(\Gamma) \supseteq \mathcal{R}_l^i(\Gamma)$ applies to any $i \in N$. As usual, we declare $\mathcal{R}_k^{-i}(\Gamma) := \times_{i\in N\setminus\{j\}}\mathcal{R}_k^i(\Gamma)$ for any $i \in N$ and $k \in \mathbb{N}$. Obviously, $\mathcal{R}_k^{-i}(\Gamma) \supseteq \mathcal{R}_l^{-i}(\Gamma)$ holds for any $i \in N$, whenever $l \geq k$ is given. Next, fix set $\mathcal{R}_\infty^i(\Gamma) := \cap_{k\in\mathbb{N}}\mathcal{R}_k^i(\Gamma)$. Because

$$\mathcal{R}_\infty(\Gamma) = \cap_{k\in\mathbb{N}}\mathcal{R}_k(\Gamma) = \cap_{k\in\mathbb{N}}\left(\times_{i\in N}\mathcal{R}_k^i(\Gamma)\right) = \times_{i\in N}\left(\cap_{k\in\mathbb{N}}\mathcal{R}_k^i(\Gamma)\right) = \times_{i\in N}\mathcal{R}_\infty^i(\Gamma)$$

is satisfied, the set $\mathcal{R}_\infty^i(\Gamma)$ is the i-th component of $\mathcal{R}_\infty(\Gamma)$. Strategies belonging to $\mathcal{R}_\infty^i(\Gamma)$ are said to be *iteratively* $\mathcal{C}$*-favorable*. Since the strategic game Γ is finite, the process of elimination stops at some round. That is, there exists a round l such that $\mathcal{R}_k(\Gamma) = \mathcal{R}_\infty(\Gamma)$ holds for any $k \geq l$. Hence, $\mathcal{R}_k^i(\Gamma) = \mathcal{R}_\infty^i(\Gamma)$ holds for any $i \in N$ and any $k \geq l$. For notational simplicity, we denote from now on the reduced strategic game $\Gamma|_{\mathcal{R}_k(\Gamma)}$ by Γ_k. Whenever $\mathcal{C}$ is a choice rule that gives for any decision problem $(\mathcal{R}_k^{-i}(\Gamma), C_{\Gamma_k}^i)$ of any player $i \in N$ at least one solution, then $\mathcal{R}_\infty(\Gamma) \neq \emptyset$ results and thus $\mathcal{R}_\infty^i(\Gamma) \neq \emptyset$ applies to any $i \in N$. The following remark summarizes the properties of the sequence $\left(\mathcal{R}_k^i(\Gamma)\right)_{k\in\mathbb{N}}$ just discussed.

Remark 2.2 *Consider a finite strategic game* Γ. *Let* $\mathcal{C}$ *be a choice rule for individual decision problems under uncertainty and define inductively the sequence* $\left(\mathcal{R}_k^i(\Gamma)\right)_{k\in\mathbb{N}}$ *of sets of strategies by* $\mathcal{R}_{k+1}^i(\Gamma) := \mathcal{R}_1^i(\Gamma|_k)$ *for any* $k \in \mathbb{N}$. *Then* *(a)* $\left(\mathcal{R}_k^i(\Gamma)\right)_{k\in\mathbb{N}}$ *is the i-th component of the process of $\mathcal{C}$-unfavorable strategy profiles.*

(b) $\mathcal{R}_k^i(\Gamma) \supseteq \mathcal{R}_l^i(\Gamma)$ *holds, if $l \geq k$ holds.*

(c) *there exists a round l such that $\mathcal{R}_k^i(\Gamma) = \mathcal{R}_\infty^i(\Gamma)$ holds for any $k \geq l$.*

(d) $\mathcal{R}_\infty^i(\Gamma) \neq \emptyset$ *holds, if and only if $\mathcal{C}(\Phi) \neq \emptyset$ holds for any individual decision problem $\Phi = \left(\mathcal{R}_j^{-i}(\Gamma), C_{\Gamma_k}^j\right)$ with $k \geq 0$ and $j \in N$.*

In the succeeding proofs we generally make use of these properties without reference. Further, to simplify the notation, we omit the argument Γ, whenever it is obvious on which strategic game the choice rule is (repeatedly) applied. In this case the sequence of sets of player i's strategies generated by the iterated application of choice rule $\mathcal{C}$ is simply denoted by $(R_k^i)_{k\in\mathbb{N}}$ and the set of all strategies of player i that survive this process by R_∞^i.

3 The Best Choice Property of IACR

In this section we demand that the players' strategies surviving IACR establish the largest family having the best choice property. The last property is a generalization of the best response property of Pearce (1984) and is defined as follows.

Definition 3.1 *Consider a finite strategic game $\Gamma := (S^i, z^i)_{i\in N}$ and a choice rule $\mathcal{C}$. A family $(\tilde{S}^i)_{i\in N}$ consisting of non-empty subsets of S^i for each player $i \in N$ has the best choice property, if for each $i \in N$ every strategy $s^i \in \tilde{S}^i$ satisfies $s^i|_{\tilde{S}^{-i}} \in \mathcal{C}(\tilde{S}^{-i}, S^i|_{\tilde{S}^{-i}})$.*

In words, a family of strategy sets satisfies the best choice property, whenever each strategy being a member of one of those sets proves to be a favorable choice among all available strategies given a state space composed by the remaining sets of that family. As mentioned above this property is a generalization of the best response property of Pearce (1984) which is used to characterize the set of rationalizable strategy profiles. Indeed, in case that the choice rule selects those strategies which maximize the expected payoff for some product probability measure on the opponents' strategies the best choice property turns into the best response property of Pearce (1984).

The following properties of a choice rule become relevant in order to characterize the solution of IACR in the way as required above.

Definition 3.2 *A choice rule $\mathcal{C}$ for decision problems under uncertainty is called*

- *non-empty, if $\mathcal{C}(\Phi) \neq \emptyset$ holds for any decision problem Φ.*
- *positively consistent, if $x \in \mathcal{C}(\Phi)$ implies $x \in \mathcal{C}(\Omega, C \setminus \{y\})$ for any decision problem $\Phi := (\Omega, C)$ whose constraint C contains the distinct acts x and y.*
- *internally consistent, if $x|_E \notin \mathcal{C}(\Phi|_E)$ implies $x|_E \notin \mathcal{C}(E, \mathcal{C}(\Phi)|_E)$ for any decision problem $\Phi := (\Omega, C)$ and any non-empty event $E \subseteq \Omega$.*
- *conditionally consistent, if $x \notin \mathcal{C}(\Phi)$ implies $x|_E \notin \mathcal{C}(\Phi|_E)$ for any decision problem $\Phi := (\Omega, C)$ and any non-empty event $E \subseteq \Omega$.*

Non-emptiness guarantees that the choice rule gives for any decision problem under uncertainty at least one solution. To avoid grappling with existence problems non-emptiness is part of the presumptions of the succeeding theorems. However, those results are still valid when this requirement is dropped. Another reason why we impose non-emptiness without concerns is that the most widely used choice rules satisfy it. Nevertheless we should to be aware that prominent choice rules like strict dominance (i.e. only that available act is selected that strictly dominates every other available act) fail this property.

Positive consistency says that if the distinct acts x and y are available and x is deemed as favorable, then x keeps favorable, when act y is canceled from the choice set. Since we restrict ourselves on finite choice set this property turns out being equivalent to the requirement, that for any decision problems $\Phi := (\Omega, \tilde{C})$ and $\tilde{\Phi} := (\Omega, \tilde{C})$, where $C \supseteq \tilde{C}$ and $x \in \tilde{C}$ is satisfied, $x \in \mathcal{C}(\Phi)$ implies $x \in \mathcal{C}(\tilde{\Phi})$. Note, this property is well-known in choice theory and is usually called Sen's property α (see Sen (1969) or Kreps (1988)). Actually, this property was already proposed in an earlier paper by Chernoff (1954) in a statistical context.

In order to understand internal consistency recall that set $\mathcal{C}(\Phi)|_E$ consists exactly of those acts which are restrictions on E of favorable acts in decision problem Φ. This property demands that an act considered as unfavorable in the reduced decision problem $\Phi|_E := (E, C|_E)$ does not become favorable when the constraint is limited to acts which are favorable in the unrestricted decision problem.

With regard to conditional consistency recall that $\Phi|_E := (E, C|_E)$ stands for the decision problem Φ reduced on the non-empty event $E \subseteq \Omega$ where its constraint is given by $C_E := \{x_E : x \in C\}$. According to this definition conditional consistency means that the restriction of an unfavorable act on a non-empty event E is unfavorable in the decision problem reduced on E. This requirement rules out cases at which acts being once considered as unfavorable become favorable, when additional information about the true state of the world is revealed.

The following table lists properties of prominent choice rules. Recall that pure strategy undominance in sense of Börgers (1993) means that an available act is favorable whenever there is a non-empty event on which this act is not weakly dominated by some other available act. The Laplace rule singles out those available acts which maximizes the expected payoff, where the same probability is assigned to each possible state. The letters E, P, I, C and M stands for non-emptiness and positive, internal, conditional and marginal consistency, respectively. Latter property is introduced in section 4. It turns out that these properties are independent, that is, none of those properties can be deduced by the other four properties (see also the examples given in section 4).

		Properties				
Choice rule		**E**	**P**	**I**	**C**	**M**
Strict dominance			×			
Strict undominance	in pure acts	×	×	×	×	×
	in mixed acts	×	×	×	×	×
Weak dominance			×			
Weak undominance	in pure acts	×	×	×		×
	in mixed acts	×	×	×		×
Börgers pure strategy undominance		×	×	×	×	×
Laplace rule		×	×			
Maximin rule		×	×			×
Minimax regret rule		×				×

Figure 2: Popular choice rules and their properties

The objective of this section is reached in two steps. At first we figure out the properties of a choice rule entailing that the solution generated by IACR satisfies the best choice property. Then we examine the converse of this statement and figure out properties of a choice rule entailing that any family of sets of strategies having the best choice property survives IACR. As we will see putting these results together we end up in the characterization of IACR as required at the beginning of this section.

Lemma 3.3 *Consider a strategic game Γ and a non-empty, internally consistent choice rule $\mathcal{C}$. Then family $(R^i_\infty)_{i\in N}$ satisfies the best choice property.*

Proof: Suppose the opposite and presume that there exists a player j having an available strategy $s^j \in R^j_\infty$ for which $s^j|_{R^{-j}_\infty} \notin \mathcal{C}\left(R^{-j}_\infty, S^j|_{R^{-j}_\infty}\right)$ holds. Recall that there is a round l such that $R^i_k = R^i_\infty$ applies to any $k \geq l$ and any player $i \in N$. Consequently, $s^j|_{R^{-j}_l} \notin \mathcal{C}\left(R^{-j}_l, S^j|_{R^{-j}_l}\right)$ holds. We show by induction that $s^j|_{R^{-j}_l} \notin$

$\mathcal{C}\left(R_l^{-j}, R_m^j|_{R_l^{-j}}\right)$ is true for any $1 \leq m \leq l$. Given $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, S^j|_{R_l^{-j}}\right)$ internal consistency implies $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, \mathcal{C}\left(S^{-j}, S^j\right)|_{R_l^{-j}}\right)$. By convention $R_1^j = \mathcal{C}\left(S^{-j}, S^j\right)$ and therefore $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, R_1^j|_{R_l^{-j}}\right)$ results. Now, suppose that $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, R_m^j|_{R_l^{-j}}\right)$ is satisfied for some $1 \leq m < l$. Applying again internal consistency we reach $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, \mathcal{C}(R_m^{-j}, R_m^j)|_{R_m^{-j}}\right)$. That is, $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, R_{m+1}^j|_{R_m^{-j}}\right)$ holds and thereby the above claim is proved. In particular, we have shown that $s^j|_{R_l^{-j}} \notin \mathcal{C}\left(R_l^{-j}, R_l^j|_{R_l^{-j}}\right)$, or equivalently $s^j \notin R_{l+1}^j$, applies. However, this conflicts with our premise $s^j \in R_\infty^j$. ■

Next we deal with the converse of the statement given in lemma 3.3. We figure out conditions on the choice rule ensuring that any strategy profile of any family having the best choice property proves to be a solution generated by iterated application of that choice rule.

Lemma 3.4 *Consider a finite strategic game $\Gamma := (S^i, z^i)_{i \in N}$ and a non-empty choice rule $\mathcal{C}$ satisfying both positive and conditional consistency. If a family $(\tilde{S}^i)_{i \in N}$ of sets of strategies has the best choice property, then each strategy profile $s \in \times_{i \in N} \tilde{S}^i$ survives the iterated application of $\mathcal{C}$ to strategic game Γ.*

Proof: Suppose the family $(\tilde{S}^i)_{i \in N}$, where $\tilde{S}^i \subseteq S^i$ holds for any $i \in N$ satisfies the best choice property. Note $\tilde{S}^i \subseteq R_\infty^i$ holds for each $i \in N$, whenever $\tilde{S}^i \subseteq R_k^i$ holds for each $k \in \mathbb{N}$ and each $i \in N$. By induction on elimination rounds we will establish the latter statement. Clearly, $\tilde{S}^i \subseteq S^i$ holds for each $i \in N$. Pick, for some player j, some strategy $s^j \in \tilde{S}^j$. By presumption, $s^j|_{\tilde{S}^{-j}} \in \mathcal{C}\left(\tilde{S}^{-j}, S^j|_{\tilde{S}^j}\right)$ is satisfied. Conditional consistency implies $s^j \in \mathcal{C}\left(S^j, S^{-j}\right)$. That means nothing but $s^j \in R_1^j$. Suppose for some elimination round k that $\tilde{S}^i \subseteq R_k^i$ applies to any $i \in N$. Choose some strategy $s^j \in \tilde{S}^j$ for some player j. Since $s^j|_{\tilde{S}^{-j}} \in \mathcal{C}(\tilde{S}^{-j}, S^j)$ holds, the induction premise together with conditional consistency leads to $s^j|_{\tilde{R}_k^{-j}} \in \mathcal{C}\left(R_k^{-j}, S^j|_{R_k^{-j}}\right)$. By positive consistency $s^j|_{\tilde{R}_k^{-j}} \in \mathcal{C}\left(R_k^{-j}, R_k^j|_{R_k^{-j}}\right)$ results. Latter means nothing different but $s^j \in R_{k+1}^j$. Since player j has been arbitrarily chosen, we have finally established that each strategy profile of $\times_{i \in N} \tilde{S}^i$ survives the iterated application of $\mathcal{C}$ to game Γ. ■

The preceding lemmata are summarized in the following theorem which gives conditions on choice rules such that every family satisfies the best choice property, if and only if each member of its product survives the iterated application of $\mathcal{C}$.

Theorem 3.5 *Consider a finite strategic game Γ and let $\mathcal{C}$ be a non-empty choice rule which satisfies positive, conditional and internal consistency. Then a strategy profile $s \in S$ survives the iterated application of choice rule $\mathcal{C}$ to Γ, if and only if there exists a family $(\tilde{S}^i)_{i \in N}$ of sets of strategies satisfying the best choice property and $s \in \times_{i \in N} \tilde{S}^i$.*

Proof: (*if*) This direction is the result of lemma 3.4. (*only if*) By lemma 3.3 family $(R^i_\infty)_{i\in N}$ satisfies the best choice property. Note that $s \in \times_{i\in N} R^i_\infty$ holds and thus we have found a family of subsets of players' strategy sets having the desired properties. ■

Consider some strategic game $\Gamma := (S^i, z^i)_{i\in N}$ and some non-empty choice rule $\mathcal{C}$ satisfying conditional and internal consistency. Let $\mathscr{X}$ be the finite class of all families having the best choice property in game Γ. As we have shown in lemma 3.3 the family $(R^i_\infty)_{i\in N}$ satisfies the best choice property. For this reason class $\mathscr{X}$ is non-empty. Suppose the two families $(X^i)_{i\in N}$ and $(Y^i)_{i\in N}$ belong to $\mathscr{X}$. It turns out that their union $(X^i)_{i\in N} \cup (Y^i)_{i\in N} := (X^i \cup Y^i)_{i\in N}$ also belongs to $\mathscr{X}$. To see that choose some $s^j \in X^j \cup Y^j$. Without loss of generality assume that $s^j \in X^j$ holds. Since family $(X^i)_{i\in N}$ has the best choice property, $s^j|_{X^{-i}} \in \mathcal{C}(X^{-i}, S^j|_{X^{-i}})$ applies. Set $Z^{-j} := \times_{i\in N\setminus\{j\}}(X^i \cup Y^i)$. By conditional consistency $s^j|_{Z^{-j}} \in \mathcal{C}(Z^{-j}, S^j|_{Z^{-j}})$ results. Because that holds for any player $j \in N$ and any strategy $s^j \in X^j \cup Y^j$ we have established that family $(X^i)_{i\in N} \cup (Y^i)_{i\in N}$ belongs to $\mathscr{X}$. In consequence, family $(X^i)_{i\in N} := \bigcup\{(Y^i)_{i\in N} : (Y^i)_{i\in N} \in \mathscr{X}\}$ is a member of $\mathscr{X}$. This family is called the *largest family in game* Γ *satisfying the best choice property*. Suppose our choice rule $\mathcal{C}$ satisfies additionally positive consistency. Due to lemma 3.4 any strategy profile of the product of this family is a solution generated by iterated application of $\mathcal{C}$ to Γ. On the other side, since according to lemma 3.3 family $(R^i_\infty)_{i\in N}$ has the best choice property, each solution surviving this elimination process is also contained in that product. Hence, the solution R_∞ of game Γ corresponds to the product of the largest family of Γ which has the best choice property.

Remark 3.6 *Consider a finite strategic game* Γ *and a non-empty choice rule* $\mathcal{C}$ *satisfying positive, conditional and internal consistency. Then the solution* R_∞ *generated by iterated application of* $\mathcal{C}$ *to* Γ *is the product of the largest family in game* Γ *having the best choice property.*

4 Common Belief of Applying the Choice Rule

An *epistemic model to a strategic game* $\Gamma := (S^i, z^i)_{i\in N}$ is a tuple $\left(\Omega, (P^i)_{i\in N}, (\sigma^i)_{i\in N}\right)$ where its items are defined as follows. The finite set Ω denotes the *state space*. Its members a called *states of the world* and represent a certain resolution of all relevant issues for the players. The mapping $P^i : \Omega \to 2^\Omega$ is the serial, transitive and euclidean *possibility correspondence of player* i assigning to each state ω the possibility set $P^i(\omega)$ that comprises all states deemed possible by i at state ω. Recall seriality means that, for any $\omega \in \Omega$, $P^i(\omega) \neq \emptyset$ is satisfied and transitivity and euclideanness implies that, for any $\omega \in \Omega$, if $\tilde\omega \in P^i(\omega)$ holds then $P^i(\tilde\omega) = P^i(\omega)$ is satisfied. The mapping $\sigma^i : \Omega \to S^i$ is called *strategy function of player* i and reveals the choice of strategy of player i for each state. It is presupposed that $\sigma^i(\tilde{\omega}) = \sigma^i(\omega)$ holds for any $\tilde{\omega} \in P^i(\omega)$ and $\omega \in \Omega$. That is, at each state player i does not err about her own choice of strategy.

An event $E \subseteq \Omega$ is believed by player i at state ω, if $P^i(\omega) \subseteq E$ applies. Let P_* the transitive closure of the possibility correspondences $(P^i)_{i\in N}$ (i.e. $\tilde{\omega} \in P_*(\omega)$ holds if and only if there is a finite sequence $(i_1, \ldots, i_m)$ in N and a finite sequence $(\omega_0, \omega_1, \ldots, \omega_m)$ in Ω such that $\omega_0 = \omega$, $\omega_m = \tilde{\omega}$ and, for every $k = 1, \ldots, m$, $\omega_k \in P^{i_k}(\omega_{k-1})$ hold.) This correspondence is called *common possibility correspondence*. Clearly, $P^i(\omega) \subseteq P_*(\omega)$ is satisfied for any $\omega \in \Omega$ and any $i \in N$. Furthermore, it turns out (see e.g. Fagin et al. (1995, Lemma 2.2.1)) that an *event E is commonly believed* (i.e. every player believes E, every player believes that every player believes E, and so on) *at state ω*, if and only if $P_*(\omega) \subseteq E$ applies. Note, if $P^i(\omega) \subseteq E$ holds for any $\omega \in E$ and any player $i \in N$ then event E is commonly believed at every state belonging to E.

Fix some choice rule $\mathcal{C}$. A player i is said to *follow* (or to *apply*) *choice rule* $\mathcal{C}$, if $\sigma^i(\omega)|_{P^i(\omega)} \in \mathcal{C}\left(P^i(\omega), S^i|_{P^i(\omega)}\right)$ holds, that is, if her choice of strategy at state ω is a favorable one according to choice rule $\mathcal{C}$ and given her uncertainty $P^i(\omega)$. A player i believes at state ω that player j follows $\mathcal{C}$, if $\sigma^j(\tilde{\omega})|_{P^j(\tilde{\omega})} \in \mathcal{C}(P^j\left(\tilde{\omega}), S^j|_{P^j(\tilde{\omega})}\right)$ applies to any state $\tilde{\omega} \in P^i(\omega)$. In case that player i believes at state ω that she herself applies the choice rule $\mathcal{C}$, then she actually applies at state ω choice rule $\mathcal{C}$. This results from the fact that she is aware about her decision (formally, $\sigma(\omega) = \sigma(\tilde{\omega})$ holds for any $\tilde{\omega} \in P^i(\omega)$) and about what she considers as possible (formally, $P^i(\omega) = P^i(\tilde{\omega})$ holds for any $\tilde{\omega} \in P^i(\omega)$). We say that there is common belief at state ω that every player applies $\mathcal{C}$, if $\sigma^j(\tilde{\omega})|_{P^j(\tilde{\omega})} \in \mathcal{C}\left(P^j(\tilde{\omega}), S^j|_{P^j(\tilde{\omega})}\right)$ is satisfied for any $\tilde{\omega} \in P_*(\omega)$ and for any player $j \in N$. Clearly, by the arguments given above, common belief of applying choice rule $\mathcal{C}$ at state ω implies that choice rule $\mathcal{C}$ is actually applied by any player at state ω.

Consider an epistemic model $\left(\Omega, (P^i)_{i\in N}, (\sigma^i)_{i\in N}\right)$ to a finite strategic game $\Gamma := (S^i, z^i)_{i\in N}$. The model is said to be *consistent with a statement about the world*, whenever the model contains a state at which this statement is satisfied. A statement *characterizes a set of $T \subseteq S$ of strategy profiles*, if the following two conditions are satisfied:

(i) (*Consistency*) If the epistemic model is consistent with the statement, then at every state satisfying this statement a strategy profile $s \in T$ is realized.

(ii) (*Existence*) For every $s \in T$ there exists an epistemic model to Γ which contains a state at which this statement is satisfied and at which strategy profile s is realized.

An *epistemic statement* is a statement referring to the beliefs of the players. An *epistemic characterization for a solution concept* is given, whenever an epistemic statement is found which characterizes, for any strategic game, the set of strategy profiles resulting from this solution concept. In this section we impose properties on the choice rule entailing that the iterated application of the choice rule is characterized by common belief of applying this rule. As our example in the introduction unveiled this coincidence is by no means guaranteed. The following property is crucial for the consistency part of the epistemic characterization of IACR.

Definition 4.1 *Choice rule $\mathcal{C}$ is called marginally consistent, if for any decision problems $\Phi := (\Omega, C)$ and $\Phi' := (\Omega', C')$, where a surjective mapping $\tau : \Omega' \to \Omega$ exists such that $C' = \{y \circ \tau : y \in C\}$ holds, $x \circ \tau \in \mathcal{C}(\Omega', C')$ implies $x \in \mathcal{C}(\Omega, C)$.*

A prominent example of a choice rule violating marginal consistency is the Laplace rule. To see that consider the decision problems $\Phi := (\Omega, C)$ and $\Phi' := (\Omega', C')$, where their state spaces are given by $\Omega := \{\alpha, \beta\}$ and $\Omega' := \{(\alpha, 1), (\alpha, 2), (\beta, 1)\}$ and their constraints by $C := \{(x_\alpha, x_\beta) := (2, 2), (y_\alpha, y_\beta) := (0, 5)\}$ and $C' := \{(x'_{(\alpha,1)}, x'_{(\alpha,2)}, x'_{(\beta,1)}) := (2, 2, 2), (y'_{(\alpha,1)}, y'_{(\alpha,2)}, y'_{(\beta,1)}) := (0, 0, 5)\}$. The mapping τ assigning to each state from Ω' the state from Ω having the same Greek letter is surjective and has the property $C' = \{y \circ \tau : y \in C\}$. Obviously, if the Laplace rule is applied decision maker chooses act $(2, 2, 2)$ for decision problem Φ' and act $(0, 5)$ for decision problem Φ. However, this contradicts marginal consistency.

The following lemma specifies properties on choice rules such that outcomes resulting from choice rule following behavior and common belief of that survive IACR.

Lemma 4.2 *Consider an epistemic model $(\Omega, (P^i)_{i\in N}, (\sigma^i)_{i\in N})$ to a strategic game $\Gamma := (S^i, z^i)_{i\in N}$. If at state ω there is common belief that each player follows a non-empty, positively, conditionally and marginally consistent choice rule, then the strategy profile $\sigma(\omega)$ survives the iterated application of choice rule $\mathcal{C}$.*

Proof: Suppose at state ω there is common belief that each player follows the non-empty, positively, conditionally and marginally consistent choice rule $\mathcal{C}$. Fix $S^i_*(\omega) := \{\sigma^i(\hat{\omega}) : \hat{\omega} \in P_*(\omega)\}$ for every $i \in N$ and set $S^{-i}_*(\omega) := \times_{j\in N\setminus\{i\}} S^j_*(\omega)$ and $S_*(\omega) := \times_{i\in N} S^i_*(\omega)$. We show that family $(S^i_*(\omega))_{i\in N}$ has the best choice property. Pick an arbitrary player $i \in N$ and an arbitrary strategy $\tilde{s}^i \in S^i_*(\omega)$. Then there exists a state $\tilde{\omega} \in P_*(\omega)$ such that $\tilde{s}^i = \sigma^i(\tilde{\omega})$ is satisfied. Since at state ω there is common belief that each player applies choice rule $\mathcal{C}$ we observe that

$$\sigma^i(\tilde{\omega}) \circ \sigma^{-i}|_{P^i(\tilde{\omega})} \in \mathcal{C}\left(P^i(\tilde{\omega}), \{s^i \circ \sigma^{-i}|_{P^i(\tilde{\omega})} : s^i \in S^i\}\right)$$

applies. Note that $P^i(\tilde{\omega}) \subseteq P_*(\omega)$ holds. Because choice rule $\mathcal{C}$ is conditionally consistent

$$\sigma^i(\tilde{\omega}) \circ \sigma^{-i}|_{P_*(\omega)} \in \mathcal{C}\left(P_*(\omega), \{s^i \circ \sigma^{-i}|_{P_*(\omega)} : s^i \in S^i\}\right)$$

results. The mapping $\tau^{-i} : P_*(\omega) \to S^{-i}_*(\omega)$ given by $\tau^{-i} := \sigma^{-i}|_{P_*(\omega)}$ is well-defined and surjective. Obviously, it holds

$$\sigma^i(\tilde{\omega})|_{S^{-i}_*(\omega)} \circ \tau^{-i} \in \mathcal{C}\left(P_*(\omega), \{s^i|_{S^{-i}_*(\omega)} \circ \tau^{-i} : s^i \in S^i\}\right) .$$

Marginal consistency implies $\sigma^i(\tilde{\omega})|_{S_*^{-i}(\omega)} \in \mathcal{C}\left(S_*^{-i}(\omega), S^i|_{S_*^{-i}(\omega)}\right)$. Thereby it has been established that family $(S_*^i(\omega))_{i \in N}$ has the best choice property. Now consider the actual state ω. Note again that $P^i(\omega) \subseteq P_*(\omega)$ holds. Because i's possibility correspondence is serial there exists a state $\tilde{\omega} \in P_*(\omega)$ belonging to $P^i(\omega)$. By presumption $\sigma^i(\omega) = \sigma^i(\tilde{\omega})$. Hence, $\sigma^i(\omega) \in S_*^i(\omega)$ is given and $\sigma^i(\omega)|_{S_*^{-i}(\omega)} \in \mathcal{C}(S_*^{-i}(\omega), S^i|_{S_*^{-i}(\omega)})$ results. By lemma 3.4 strategy $\sigma^i(\omega)$ survives the iterated application of choice rule $\mathcal{C}$. ■

As shown in table 2 the maximin rule violates conditional consistency. The example given in the introduction revealed that not any strategy profile resulting from maximin rule following behavior and common belief of that survives the iterated application of the maximin rule. Indeed, as we will show in the following three examples, none of the consistency properties mentioned in lemma 4.2 can be omitted without causing such divergency between the solutions generated by IACR and by the decision-theoretic assumptions of choice rule following behavior and common belief of that. Note, the choice rules considered in these examples satisfy all but one of the properties introduced in this paper.

Example 4.3 *Consider the choice rule which selects those available acts yielding at least at one state at least the average payoff over the available acts at this state. It turns out that this choice rule fails positive consistency but satisfies all other properties. The strategic game Γ_2 is solved by iterated application of this choice rule. At the first round strategy d is canceled, at the second round strategy m and at the third round strategy r. Consequently, IACR results in the strategy profile (u, l). But as we will see other profiles are consistent with choice rule following behavior and common belief of that.*

		Player C	
		l	r
Player R	u	$(3,1)$	$(3,0)$
	m	$(2,0)$	$(2,1)$
	d	$(0,1)$	$(0,1)$

Figure 3: Strategic Game Γ_2

Consider the epistemic model, where the state space consists only of state α and the strategy functions are defined by $\sigma^R(\alpha) := m$ and $\sigma^C(\alpha) := r$. Note, given the possibility set $P^R(\alpha) = \{\alpha\}$, at state α player R acts as if she follows the above choice rule. Furthermore, at state α player C acts as if he also follows this choice rule. Therefore at state α strategy profile (m, r) is realized, both players act according to the above choice rule and that is commonly believed among them. Consequently, strategy profiles which do not survive the IACR can occur when the players obey to that choice rule and *have common belief of that behavior.*

Example 4.4 *Suppose the strategic game depicted below is solved by iterated application of the choice rule of weak undominance in pure acts. Note that this choice rule satisfies all properties with exception of conditional consistency. Clearly, strategy profile (u, l) is the unique solution of IACR. The question is whether in any situation in which players act according to the weak undominance rule and having the common belief of that this strategy profile is realized.*

Player C

		l	r
	u	$(1,1)$	$(0,0)$
Player R	d	$(0,0)$	$(0,0)$

Figure 4: Strategic Game Γ_3

Consider the epistemic model, where the state space consists only of state α and the strategy functions are defined by $\sigma^R(\alpha) := d$ and $\sigma^C(\alpha) := r$. Obviously, at state α strategy profile (d, l) is realized, both players act according to the weak undominance rule and that is commonly believed among them. Consequently, strategy profiles outside the solution generated by IACR are consistent with choice rule following behavior and common belief of that.

Example 4.5 *Consider decision makers who delight in the lucky number 13. They consider any act as favorable which is strictly undominated or which yields the number 13 at two or more possible states of the world. This choice rule satisfies all properties except that of marginal consistency. Suppose that this choice rule is repeatedly applied on the strategic game depicted below. It turns out that only strategy profile (u, l) survives this elimination process.*

Player C

		l	r
	u	$(14,1)$	$(1,0)$
Player R	d	$(13,1)$	$(0,0)$

Figure 5: Strategic Game Γ_4

Although strategy profile (d, l) is deleted at the first round, it is consistent with choice rule following behavior and common belief of that. To see that supplement the game with the epistemic model consisting of state space $\Omega := \{\alpha, \beta\}$, possibility correspondences $P^i(\alpha) := P^i(\beta) := \{\alpha, \beta\}$ for any player $i \in \{C, R\}$ and strategy functions $\sigma^R(\alpha) :=$

$\sigma^R(\beta) := d$ and $\sigma^C(\alpha) := \sigma^C(\beta) := l$, respectively. It can be easily checked that at state α strategy profile (d, l) is realized, both players act according to the above choice rule and that is commonly believed among them.

Next lemma specifies properties on choice rules implying that IACR leads to outcomes consistent with choice rule following behavior and common belief of that.

Lemma 4.6 *Consider a strategic game $\Gamma := (S^i, z^i)_{i\in N}$ and a non-empty, internally consistent choice rule $\mathcal{C}$. Then for any strategy profile s surviving the iterated application of choice rule $\mathcal{C}$ there exists an epistemic model $(\Omega, (P^i)_{i\in N}, (\sigma^i)_{i\in N})$ containing a state ω at which $\sigma(\omega) = s$ holds and there is common belief that each player follows this rule.*

Proof: Let $R_\infty := \times_{i\in N} R^i_\infty$ be the solution generated by iterated application of choice rule $\mathcal{C}$. By lemma 3.3 family $(R^i_\infty)_{i\in N}$ has the best choice property. That is, for every player $i \in N$ and every strategy $s^i \in R^i_\infty$ it holds $s^i|_{R^{-i}_\infty} \in \mathcal{C}\left(R^{-i}_\infty, S^i|_{R^{-i}_\infty}\right)$. Fix state space $\Omega := R_\infty$ and let strategy function σ^i of any player i be the projection of R_∞ on R^i_∞. The player i's possibility correspondence P^i is given by $P^i(\omega) := \{\sigma^i(\omega)\} \times R^{-i}_\infty$. It turns out that tuple $(\Omega, (P^i)_{i\in N}, (\sigma^i)_{i\in N})$ is an epistemic model. As remarked above at every state each player follows choice rule $\mathcal{C}$ and therefore at every state there is common belief of that. Thereby we have established that for every strategy profile surviving the iterated application of choice rule $\mathcal{C}$ an epistemic model can be constructed that contains a state at which this strategy profile is realized and there common belief that every player follows this rule. ■

Recall that the maximin rule fails internal consistency. As shown in the introduction this failure entails that not any strategy profile surviving the iterated application of the maximin rule is consistent with maximin rule following behavior and common belief of that. This divergency can also be observed, if the choice rule satisfies all properties discussed in this paper except that of internal consistency. In the following example we specify such a choice rule.

Example 4.7 *Consider decision makers fearing the unlucky number 13. They avoid acts yielding at every state the number 13. Whenever possible, they ignore such acts and pick only acts which are strictly undominated among the remaining available acts. It can be easily checked that this rule satisfies all the properties introduced in this paper except that of internal consistency. Suppose now that this choice rule is repeatedly applied on strategic game Γ_5. Obviously, the solution generated by this deletion process is strategy profile (u, l).*

Is this solution consistent with choice rule following behavior and common belief of that? In case that player R deems only states as possible at which player C chooses l her strategy u induces an act yielding at every possible state the unlucky number 13 and, thus, is considered as unfavorable. If R considers as possible that C chooses r then act u becomes the

		Player C	
		l	r
	u	$(13, 1)$	$(14, 0)$
Player R	d	$(12, 1)$	$(12, 0)$

Figure 6: Strategic Game Γ_5

favorable act (since it does not yield at every state the unlucky number and strictly dominates act d). However, note that R also believes that C employs the same choice rule. Obviously, no matter what C thinks about the choice of R he will never select l according to this rule. Therefore strategy u is not realized under common belief of applying this choice rule.

Bringing together lemmata 4.2 and 4.6 we obtain the following characterization result of IACR.

Theorem 4.8 *Suppose the non-empty choice rule $\mathcal{C}$ is positively, internally, conditionally and marginally consistent. Then iterated application of $\mathcal{C}$ is characterized by common belief that each player follows this rule.*

As demonstrated in the examples of this section this result is weak in the following sense. None of the consistency properties can be canceled without breaking down the epistemic characterization.

5 Concluding Remarks

In this paper we specified conditions on choice rules such that the solutions obtained by the iterated application of the choice rule coincide with the solutions resulting from common belief of choice rule following behavior. Thereby the solution concept of IACR is equipped with an epistemic motivation. As we have shown this equivalence holds, if the non-empty choice rule satisfies positive, internal, conditional and marginal consistency. Examples of choice rules which satisfy these properties are strict undominance, strict undominance in mixtures and Börgers' undominance concept. We demonstrated that none of the consistency properties can be canceled without demolishing the above epistemic characterization. But note that does not mean that no weaker version of our consistency axioms can be found which also entails this characterization. Figuring out weaker conditions could be an objective of future research.

Reviewing our analysis we can break up our central theorem 4.8 into two parts. If the non-empty choice rule satisfies internal consistency, then every strategy profile surviving IACR is consistent with the assumption of common belief of choice rule following behavior (see lemma 4.6). This result is relevant e.g. for the choice rules of weak undominance or weak undominance in mixtures which fulfill internal consistency but fail conditional consistency. However, as shown in example 4.7, internal consistency does not ensure that any outcome under common belief of choice-rule following behavior survives IACR. Latter is guaranteed if the non-empty choice rule satisfies positive, conditional and marginal consistency (see lemma 4.2).

Lastly, some remarks to my framework: (*Strategic game*) In my definition of a strategic game each player is associated with a payoff function assigning to each strategy profile a real number which could be interpreted as a monetary payoff given to that player. I favored such strategic game form since the most widely used choice rules are based on numerical comparison. But this form is by no means necessary. A more general formulation where the individual payoff functions are substituted by a single outcome function which assigns to each strategy profile a certain outcome is possible. I think such generalization would allow to consider choice rules which incorporate ideas of fairness and altruism. Furthermore, to circumvent mathematical difficulties, I supposed that the players' strategy sets are finite. Clearly, since many strategic games of interest have infinite strategy sets, this restriction should be removed in future work. But I suggest that the nature of my results is not affected by such generalization. (*Epistemic model*) The epistemic model I employed in this paper differs from that currently used in standard epistemic game theory. There Harsanyi type spaces (see Harsanyi (1967/68)) are taken as basis to describe the belief hierarchies of the players. It causes no difficulties to transform our epistemic model in such type spaces. There is only one point to consider. Unlike the original version each type has to be associated with a possibility set instead of a probability measure. The reason why I preferred the epistemic model based on an information structure (or Kripke structure, as it is called in modal logic) is to underscore the qualitative nature of my approach. (*Possibility correspondence*) For the information structure, I presupposed that the possibility correspondence of the player's are serial, transitive and euclidean. Latter two assumptions entail that the decision maker has the ability of introspection (i.e. if she believes something then she believes that she believes it and if she does not belief something she believes that she does not believe it). It turns out that these two assumptions are superfluous. All results are still valid, if they are omitted. We introduced them to make our theorems compatible to the stories told in the introduction and in the examples. If we abandon transitiveness and euclideanness these stories has to be modified. We have to omit the assumption that every player follows the choice rule and substitute it by the assumptions every player believes that every player (including herself) follows this choice rule (which is implicit in our common belief assumption). This modification works, but the stories become more intricate.

References

Bernheim, D. (1984). Rationalizable Strategic Behavior. *Econometrica*, 52:1007–1028.

Börgers, T. (1993). Pure Strategy Dominance. *Econometrica*, 61:423–430.

Chernoff, H. (1954). Rational Selection of Decision Functions. *Econometrica*, 22:423–443.

Fagin, R., Halpern, J., Moses, Y., and Vardi, M. (1995). *Reasoning about Knowledge*. MIT Press, Cambridge.

Halpern, J. and Pass, R. (2009). Iterated Regret Minimization: A New Solution Concept. In Boutilier, C., Ed., *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI 2009)*, 153–158.

Harsanyi, J. (1967/68). Games with Incomplete Information Played by Bayesian Players, Parts I-III. *Management Science*, 14:159–182, 320–334, 486–502.

Kreps, D. (1988). *Notes on the Theory of Choice*. Westview Press, Boulder.

Niehans, J. (1948). Zur Preisbildung bei ungewissen Erwartungen. *Scbweizerische Zeitschrift für Volkswirtschaft und Statistik*, 84.

Pearce, D. (1984). Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica*, 52:1029–1050.

Savage, L. (1951). The Theory of Statistical Decision. *Journal of the American Statistical Association*, 46:55–67.

Savage, L. (1954). *The Foundations of Statistics*. Wiley, New York.

Sen, A. (1969). Quasi-transitivity, Rational Choice, and Collective Decisions. *Review of Economic Studies*, 36:381–393.