The non-existence of a universal topological type space*

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Abstract

The concept of types was introduced by Harsányi [8]. In the literature there are two approaches for formalizing types, type spaces: the purely measurable and the topological models. In the former framework Heifetz and Samet [11] showed that the universal type space exists and later Meier [13] proved that it is complete. In this paper we examine the topological approach and conclude that there is no universal topological type space in the category of topological type spaces.

1 Introduction

In incomplete information situations the question of what the players believe about the given situation, and what the players believe about the other players' beliefs about the situation and so on, is a cardinal one. The explicit appearance of hierarchies of beliefs[1], however, can make the analysis extremely difficult.

By introducing the concept of type, Harsányi [8] avoided the explicit appearance of hierarchies of beliefs. His approach can be summarized very briefly and roughly as follows: substitute the hierarchies of beliefs with types, collect all types into an object, and let the probability measures on this object be the players' (subjective) beliefs. Henceforth, we call this approach *Harsányi program*.

Two questions arise, however, in relation with the Harsányi program. (1) Is the concept of type itself appropriate for the proposal under consideration? (2) Can every hierarchy of beliefs be represented as a type?

Question (1) consists of two subquestions: (A) Can all types be collected into one object? The concept of the universal type space introduced by Heifetz and Samet [11] formalizes this requirement: the universal type space in a certain category of type spaces is a type space that (a) it is in the given category, and

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[1] In this paper we use the terminology hierarchy of beliefs instead of coherent hierarchy of beliefs.

(b) every type space of the given category can be mapped into it in a unique way. In other words, the universal type space is the most general type space, it contains all type spaces (all types).

(B) Can every probability measure on the object of the collected types be a (subjective) belief? Brandenburger [5] introduced the notion of a complete type space: a type space is complete, if the type functions in it are onto. To put it differently, a type space is complete, if every probability measure on the object consisting of the types of the model is assigned to a type.

Question (2) is on whether or not the universal type space contains every hierarchy of beliefs. Mathematically speaking a hierarchy of beliefs defines an inverse system of measure spaces; so this question can be rephrased as follows: do the considered inverse systems of measure spaces have inverse limits?

Roughly speaking two formalizations of type space appear in the literature[2]: the purely measurable and the topological. In the former, every concept is a measure theoretic one, there is no topology in this approach. In the latter, all concepts are topological. Table 1 lists the main features of the two models.

	purely measurable model	topological model
parameter space	measurable space	topological space
type space	measurable space	topological space
the class of events	the σ-field of a measurable space	the Borel σ-field of a topological space
type function	measurable function	continuous function
beliefs	probability measures	(regular) probability measures
type morphism	measurable function	continuous function

Table 1: Type spaces

Compare the two approaches from the viewpoint of the Harsányi program. In the purely measurable framework, Heifetz and Samet proved that the universal type space exists and is unique. Quite recently Meier [13] showed that the purely measurable universal type space is complete. To sum up, in this case the answer for question (1) is affirmative, i.e., in the purely measurable framework the complete universal type space exists. In this framework, however, question (2) is an open problem.

Question (2), as we have already mentioned, is the following: every hierarchy of beliefs defines an inverse system of measure spaces, do these inverse systems of measure spaces have inverse limits? The Kolmogorov Extension Theorem is about this problem, however, it calls for topological concepts, e.g. compact regular probability measures. Therefore up to now, all papers on question (2) – e.g. Mertens and Zamir [17], Brandenburger and Dekel [6], Heifetz [9], Mertens et al. [16] – employed topological type spaces.

The above papers give affirmative answer to question (2), in other words, although they use different models, it is common in their results that their topological type spaces contain all "considered" hierarchies of beliefs. By "con-

[2]Pintér's [19] type space is an exception, that is neither topological nor purely measurable, that is mixed. In that type space the parameter space is purely measurable and the type sets are topological.

sidered" we mean that the above papers define various classes of hierarchies of beliefs, which are different from each other in the applied topological assumptions and only those hierarchies of beliefs are considered.

In order to see how the topological results are related to question (1), first we have to clarify a basic rationale of the models.

There is some minimal information that every type space must reflect, i.e. there is a certain class of events that they should contain. In incomplete information situations it is necessary to handle events like player i believes with probability at least p that an event occurs (beliefs operator see e.g. Aumann [1]). Heifetz and Samet formalize this requirement in the following way: let $(X, \mathcal{M})$ be an arbitrary measurable space, and let $\Delta(X, \mathcal{M})$ denote the set of all probability measures on it. The σ-field $\mathcal{A}$ on $\Delta(X, \mathcal{M})$ meets condition (P), if $\forall A \in \mathcal{M}$ and $\forall p \in [0,1]$:

$$\text{(P)} \qquad \{\mu \in \Delta(X, \mathcal{M}) \mid \mu(A) \geq p\} \in \mathcal{A} \quad .$$

In the purely measurable framework condition (P) determines a unique minimal σ-field, the coarsest σ-field among the σ-fields that meet condition (P).

In the topological approach we can require the following: let (X, τ) be an arbitrary topological space and denote by $B(X, \tau)$ the Borel σ-field of (X, τ). Then let $(\Delta(B(X, \tau)), \tau_*)$ be such a topological space that $B(\Delta(B(X, \tau)), \tau_*)$ meets (P).

In general, condition (P) does not determine a unique minimal τ_*, i.e. there is no weakest topology among the topologies whose Borel σ-fields meet (P), hence there is some freedom in choosing the topology of the sets of probability measures. Therefore we can conclude that question (1) is an open question in the topological framework.

	purely measurable model	topological model
Question (1)	$\surd$	?
Question (2)	?	$\surd$

Table 2: Answers for the Questions I.

Table 2, which summarizes the above discussion, shows an interesting duality. While in the purely measurable framework question (1) is answered affirmatively and question (2) is an open question, in the topological framework question (1) is open and question (2) is answered affirmatively (the topological type spaces of Mertens and Zamir, Brandenburger and Dekel, Heifetz, Mertens et al. contain all considered hierarchies of beliefs).

In this paper, we consider question (1) in the topological framework. In order to do so, we define the category of topological type spaces. Our main result (Theorem 3.1) argues that there is no universal topological type space in the category of topological type spaces. In other words, we give a negative answer to question (1) in the topological framework; we conclude that there is no universal topological type space, i.e. the Harsányi program breaks down on the topological path, see Table 3.

Some comments on our result. The observation that there is no weakest topology among the topologies whose Borel σ-fields meet property (P) is responsible for our negative result.

	purely measurable model	topological model
Question (1)	$\surd$	$\emptyset$
Question (2)	?	$\surd$

Table 3: Answers for the Questions II.

Dropping the Hausdorff property of the considered topological spaces is not strange at all. In the purely measurable approach, generally, the singleton sets are not events. Therefore any generalization in this direction is not unusual. Furthermore, in the proof of our main result (Theorem 3.1) the three topological type spaces are based on a finite parameter space, and their Borel σ-fields coincide with the Borel σ-fields of Mertens and Zamir's, Brandenburger and Dekel's, Heifetz's, Mertens et al.'s models. We emphasize that it is not the lack of the Hausdorff property that is responsible for our negative result.

It is also worth noting that every topological type space, considered in this paper, can be mapped uniquely by a measurable type morphism into the purely measurable universal type space. In this sense, the purely measurable type space is rich enough, it contains every topological type space. Therefore, we can interpret our negative result in the following way: the purely measurable universal type space cannot be topologized in a way that it be a universal topological type space.

The paper is organized as follows. In the next section, we discuss the basic notions and properties of topological type spaces. Section 3 discusses our main result. The last section concludes briefly.

2 The topological type space

First of all some notations: $\#N$ is for the cardinality of the set N. Furthermore, let (X, τ) be an arbitrary topological space. $B(X, \tau)$ is for the Borel σ-field of (X, τ). $\Delta(B(X, \tau))$ denotes the set of probability measures on the σ-field $B(X, \tau)$. Finally, δ_x is for the Dirac measure concentrated at point x.

Next, we focus on the topological type space.

Assumption 2.1. *The parameter space (S, τ_S) is an arbitrarily fixed topological space.*

We do not assume that (S, τ_S) is either compact, Polish or Hausdorff, it can be an arbitrary topological space.

Definition 2.2. *Let Ω be the space of states of the world, N be the set of the players*[3]*, w.l.o.g. we can assume that $0 \notin N$, and let $N_0 \doteq N \cup \{0\}$. Furthermore, $\forall i \in N_0$: let τ_i be a topology on Ω. Topology τ_i represents player i's information, τ_0 is the information available for Nature, hence it is the representative of τ_S. Moreover, let $\tau_\Omega \doteq \bigvee_{i \in N_0} \tau_i$, where $\vee$ is for the coarsest common refinement of the topologies under consideration.*

[3]We do not impose any restriction on the set of the players, that may have an arbitrary cardinality and no structure at all.

Every point in Ω provides a complete description of the actual state of the world, i.e. it includes both the state of Nature and the players' states of minds. The different topologies are for modeling the players' information, they have the same role as in e.g. Aumann's [1] paper the partitions have. Therefore, if $\omega, \omega' \in \Omega$ are not topologically distinguishable [4] in the topology τ_i then player i is not able to discern the difference between them, i.e. she knows and believes the same things, and behaves in the same way at the two states ω and ω'. τ_Ω represents all the information available in the model, it is the topology we get by pooling the information of the players and Nature.

In general a topology captures robustness, stability, approximation, continuity and convergence. Moreover, the previous papers on type spaces which use topology do so because they want to apply the Kolmogorov Extension Theorem, not because this is the natural mathematical apparatus to express information. However, even if a certain topology is taken because of only mathematical reasons, since the events in the model are the Borel sets of the given topology, the chosen topology determines, describes and represents the information of the considered player.

Definition 2.3. *Let $\{(\Omega, \tau_i)\}_{i \in N_0}$ be the set of states of the world (see Definition 2.2). The topological type space based on the parameter space (S, τ_S) is the following tuple*

$$((S, \tau_S), \{(\Omega, \tau_i)\}_{i \in N_0}, (\Delta(B(\Omega, \tau_\Omega)), \tau_*), g, \{f_i\}_{i \in N}) \ ,$$

where

1. $g : \Omega \to S$ *is* τ_0*-continuous,*
2. $\forall i \in N$*: player* i*'s type function* $f_i : \Omega \to (\Delta(B(\Omega, \tau_\Omega)), \tau_*)$ *is* τ_i*-continuous,*
3. $B(\Delta(B(\Omega, \tau_\Omega)), \tau_*)$ *meets property* (P) .

The above defined type space is not a Harsányi type space (see Heifetz and Mongin [10]). The Harsányi type space is such a type space that meets the points *1.*, *2.*, *3.* and

4. $\forall \omega \in \Omega$, $\forall i \in N$: $f_i(\omega)|_{B(\Omega, \tau_i)} = \delta_\omega$.

Since all our results remain valid for the Harsányi type space, and the formalism of the "general" type space is more simple, we do not discuss the Harsányi type space separately in this paper.

The above definition of the topological type space has "degree of freedom one:" τ_*. The only thing we require is that the Borel σ-field of τ_* meets property (P), hence we keep the model as general as possible.

The topological type space of Definition 2.3 differs from Heifetz and Samet's [11] type space in three main points. First, our approach is topological, while Heifetz and Samet's is purely measurable. We modified their purely measurable

[4] Let (X, τ) be arbitrarily fixed topological space and $x, y \in X$ be also arbitrarily fixed. x and y are topologically indistinguishable if they have exactly the same neighborhoods; i.e. for all open sets $U \in \tau$, we have $x \in U$ if and only if $y \in U$. See axiom T_0.

concept in such a way that the σ-fields are the Borel σ-fields, and require the type functions to be continuous and not only measurable.

Second, we do not use the Cartesian product of the parameter space and the type sets, we refer only to the topologies. By following strictly Heifetz and Samet's paper, if we take the Cartesian product of the parameter space and the type sets, and define the σ-fields (here topologies) as the σ-fields induced by the coordinate projections (e.g. τ_0 is induced by the coordinate projection $pr_0 : S \times \prod_{i \in N} T_i \to S$, see their paper for the notations) then we get to the concept of Definition 2.3. To sum up, all intuitions about type spaces - they discussed - are retained.

Moreover, since we do not use the Cartesian product, we have to connect the parameter space into the type space in some way. For doing so we use g (Mertens and Zamir [17] use a similar formalization), hence g and pr_0 have the same role in the two formalizations, in ours and in Heifetz and Samet's respectively.

Definition 2.4. *The type morphism between topological type spaces*

$$((S, \tau_S), \{(\Omega, \tau_i)\}_{i \in N_0}, (\Delta(B(\Omega, \tau_\Omega)), \tau_*), g, \{f_i\}_{i \in N})$$

and

$$((S, \tau_S), \{(\Omega', \tau'_i)\}_{i \in N_0}, (\Delta(B(\Omega', \tau_{\Omega'})), \tau'_*), g', \{f'_i\}_{i \in N})$$

$\varphi : \Omega \to \Omega'$ *is such a* τ_Ω*-continuous function that*

1. *Diagram* (1) *is commutative, i.e.* $\forall \omega \in \Omega$*:* $g(\omega) = g' \circ \varphi(\omega)$*,*

$$\begin{tikzcd} \Omega \arrow[d, "\varphi"'] \arrow[dr, "g"] & \\ \Omega' \arrow[r, "g'"] & S \end{tikzcd} \tag{1}$$

2. *Diagram* (2) *is commutative, i.e.* $\forall i \in N$, $\forall \omega \in \Omega$*:* $f'_i \circ \varphi(\omega) = \hat{\varphi} \circ f_i(\omega)$*,*

$$\begin{tikzcd} \Omega \arrow[r, "f_i"] \arrow[d, "\varphi"'] & (\Delta(B(\Omega, \tau_\Omega)), \tau_*) \arrow[d, "\hat{\varphi}"'] \\ \Omega' \arrow[r, "f'_i"] & (\Delta(B(\Omega', \tau_{\Omega'})), \tau'_*) \end{tikzcd} \tag{2}$$

where $\hat{\varphi} : (\Delta(B(\Omega, \tau_\Omega)), \tau_*) \to (\Delta(B(\Omega', \tau_{\Omega'})), \tau'_*)$ *is defined as follows:* $\forall \mu \in \Delta(B(\Omega, \tau_\Omega))$, $\forall A \in B(\Omega', \tau_{\Omega'})$*:* $\hat{\varphi}(\mu)(A) = \mu(\varphi^{-1}(A))$[5]*.*

φ *is a type isomorphism, if* φ *is a homeomorphism, and both* φ *and* φ^{-1} *are type morphisms.*

[5]It is worth noting that $\hat{\varphi}$ can be neither continuous nor measurable. Furthermore, if we impose either continuity or measurability on $\hat{\varphi}$ then our main result remains valid.

The above definition is a slight modification of Heifetz and Samet's. Naturally, our type morphism is continuous and handle the parameter space differently from their purely measurable type space. Conceptually, our definition coincides with theirs.

Corollary 2.5. *The class of topological type spaces (see Assumption 2.1 and Definition 2.3) as objects and the type morphism (see Definition 2.4) as morphism form a category. We call this category the category of topological type spaces and denote it by $\mathcal{C}_T^S$.*

By applying category theory we can collect the topological type spaces into a category (an abstract class) and examine them in a more comprehensive way.

Remark 2.6. It is worth noting that the category of topological type spaces $\mathcal{C}_T^S$ considered in this paper is more general (broader) than Mertens and Zamir's, Brandenburger and Dekel's [6], Heifetz's [9] and Mertens et al.'s [16] categories.

We discuss two properties of the topological type spaces. The first one is universality.

Definition 2.7. *A topological type space*

$$((S,\tau_S),\{(\Omega^*,\tau_i^*)\}_{i\in N_0},(\Delta(B(\Omega^*,\tau_\Omega^*)),\tau_*^*),g^*,\{f_i^*\}_{i\in N}) \tag{3}$$

is a universal topological type space in category $\mathcal{C}_T^S$, if for any topological type space of category $\mathcal{C}_T^S$

$$((S,\tau_S),\{(\Omega,\tau_i)\}_{i\in N_0},(\Delta(B(\Omega,\tau_\Omega)),\tau_*),g,\{f_i\}_{i\in N}) \tag{4}$$

there exists a unique type morphism φ from (4) to (3).

The above definition is again a simple translation of Heifetz and Samet's. The universal topological type space is the most general topological type space (object) in category $\mathcal{C}_T^S$. The continuity of the type morphism implies that the topology τ_Ω^* in a universal topological type space must be as weak as possible.

Corollary 2.8. *A universal type space is a terminal (final) object in the type space category.*

Proof. It follows from Definition 2.7. Q.E.D.

The above corollary is a transplantation of the concept of universal topological type space into category theory.

Corollary 2.9. *A universal type space is unique up to type isomorphism.*

Proof. Every terminal object is unique up to isomorphism. Q.E.D.

Heifetz and Samet proved the existence of a purely measurable universal type space which, as the above corollary indicates, is unique.

Definition 2.10. *A topological type space is complete if $\forall i \in N$: player i's type function is surjective (onto).*

Brandenburger [5] introduced the concept of a complete type space. Roughly speaking, a topological type space is complete, if for any player all her possible subjective beliefs in the given type space are types. Meier [13] showed that in the purely measurable framework the universal type space is complete.

3 The non-existence result

This section is on our non-existence result.

Theorem 3.1. *If (S,τ_S) is trivial, i.e. $\tau_S = \{\emptyset, S\}$, then there is a complete universal topological type space in category $\mathcal{C}^S_T$. However, if (S,τ_S) is not trivial then there is no universal topological type in category $\mathcal{C}^S_T$.*

Proof. If (S,τ_S) is trivial then

$$((S,\tau_S), \{(S,\tau_i)\}_{i\in N}, (\Delta(B(S,\tau_S)),\tau), id_S, \{f_i\}_{i\in N}) \ ,$$

is a complete universal type space in $\mathcal{C}^S_T$, where $\forall i \in N_0$: $\tau_i \circeq \tau_S$, $\#\Delta(B(\Omega,\tau_S)) = 1$, hence $\forall i \in N$: f_i and τ are well-defined.

If (S,τ_S) is not trivial then it is enough to examine the following topological type spaces.

Let $(S,\tau_S) \circeq (\{x,y\},\tau_d)$, where τ_d is for the discrete topology, $N \circeq \{1\}$, i.e. there is only one player, and

- $\Omega \circeq S \times [0,1]$,
- $pr_0 : \Omega \to S$, $pr_1 : \Omega \to [0,1]$,
- $\tau^u_0 \circeq \tau^l_0 \circeq \tau^E_0$ is induced by pr_0, τ^u_1, τ^l_1 and τ^E_1 are induced by the upper, the lower semicontinuity topology and the Euclidean topology on $[0,1]$ by pr_1 respectively, i.e. τ^u_1 is the coarsest topology for which pr_1 is continuous, where the subbase of the topology of $[0,1]$ consists of the sets $[0,a)$, $a \in [0,1]$, τ^l_1 is the coarsest topology for which pr_1 is continuous, where the subbase of the topology of $[0,1]$ consists of the sets $(a,1]$, $a \in [0,1]$, and τ^E_1 is the coarsest topology for which pr_1 is continuous, where the topology of $[0,1]$ is equipped by the Euclidean topology,
- $\tau^u \circeq \bigvee_{i\in N_0} \tau^u_i$, $\tau^l \circeq \bigvee_{i\in N_0} \tau^l_i$ and $\tau^E \circeq \bigvee_{i\in N_0} \tau^E_i$.

Consider the following type spaces:

$$((S,\tau_S), \{(\Omega,\tau^u_i)\}_{i\in N_0}, (\Delta(B(\Omega,\tau^u)),\tau^u_*), pr_0, f^u_1) \ , \tag{5}$$

$$((S,\tau_S), \{(\Omega,\tau^l_i)\}_{i\in N_0}, (\Delta(B(\Omega,\tau^l)),\tau^l_*), pr_0, f^l_1) \ , \tag{6}$$

and

$$((S,\tau_S), \{(\Omega,\tau^E_i)\}_{i\in N_0}, (\Delta(B(\Omega,\tau^E)),\tau^E_*), pr_0, f^E_1) \ , \tag{7}$$

where

- $\forall \omega \in \Omega$: $f^u_1(\omega) \circeq f^l_1(\omega) \circeq f^E_1 \circeq \mu(pr_1(\omega)) \times \delta_{pr_1(\omega)}$, where $\mu(pr_1(\omega)) \in \Delta(B(S,\tau_S))$ such that $\mu(\{x\}) = pr_1(\omega)$,
- τ^u_* is the finest topology on $\Delta(B(\Omega,\tau^u))$ that f^u_1 is continuous, similarly τ^l_* is the finest topology on $\Delta(B(\Omega,\tau^l))$ that f^l_1 is continuous, and τ^E_* is the finest topology on $\Delta(B(\Omega,\tau^E))$ that f^E_1 is continuous. A simple calculation shows that $B((\Delta(B(\Omega,\tau^u)),\tau^u_*))$, $B((\Delta(B(\Omega,\tau^l)),\tau^l_*))$ and $B((\Delta(B(\Omega,\tau^E)),\tau^E_*))$ meet property (P).

Therefore, (5), (6) and (7) are topological type spaces in category $\mathcal{C}_T^S$.

First, notice that the topological type spaces (5) and (6) are not type isomorphic. Moreover, $\varphi_{uE} \triangleq id_\Omega$ and $\varphi_{lE} \triangleq id_\Omega$ are type morphisms from (7) to (5) and (6) respectively[6].

Second, indirectly assume that

$$((S, \tau_S), \{(\Omega^*, \tau_i^*)\}_{i \in N_0}, (\Delta(B(\Omega^*, \tau_{\Omega^*})), \tau^*), g^*, f_1^*) \tag{8}$$

is a universal topological type space in category $\mathcal{C}_T^S$, and φ_{*u}, φ_{*l} are the two type morphisms from topological type spaces (5) and (6) to the universal topological type space (8) respectively.

Consider two cases: (1) $\forall \omega \in \Omega$: $\varphi_{*u}(\omega) = \varphi_{*l}(\omega)$. Then $\forall O \in \tau_{\Omega^*}$ open sets such that $O \cap \varphi_{*u}(\Omega) \neq \emptyset$: $\varphi_{*u}^{-1}(O) \in \tau_0^u$ and $\varphi_{*l}^{-1}(O) \in \tau_0^l$. However, then the range of $f_1^*|_{\varphi_{*u}(\Omega)}$ consists of at most two points which is a contradiction.

(2) $\exists \omega \in \Omega$: $\varphi_{*u}(\omega) \neq \varphi_{*l}(\omega)$. Then $\varphi_{*u} \circ \varphi_{uE}$ and $\varphi_{*l} \circ \varphi_{lE}$ are two different type morphisms from (7) to (8), which is a contradiction (see Definition 2.7).

In conclusion, there is no terminal object in category $\mathcal{C}_T^S$. Q.E.D.

It is worth noticing that if we consider the purely measurable approach, i.e. where the type functions and the type morphisms are measurable mappings (see Heifetz and Samet [11]) then the above counterexample does not work. In that case, the type spaces (5) and (6) are type isomorphic and both are universal in the category of the purely measurable type spaces.

4 Conclusion

Our main result Theorem 3.1 states that there is no universal topological type space in the category of topological type spaces. Moreover, it is important to note that if we consider more than one player and restrict the class of (hierarchies of) beliefs to those which meet the conditions of a Kolmogorov Extension Theorem type theorem[7] then we get the same non-existence result.

Furthermore, we emphasize again that our result does not cancel Heifetz and Samet's [11] result, it "only" shows that the purely measurable complete universal type space cannot be topologized in a way that it be a universal topological type space.

To sum up, we conclude that in the topological framework the Harsányi program breaks down, and the question remains open whether in the purely measurable framework it works or not.

[6] Notice that both $\hat{\varphi}_{uE}$ and $\hat{\varphi}_{lE}$ are continuous.

[7] I.e. we can apply the following corollary of Metivier's [18] (3.2. Theoreme pp. 269-270.) result: Let $(((X_n, \tau_n), B(X_n, \tau_n), \mu_n), \mathbb{N}, f_{mn})$ be such an inverse system of probability measures spaces that $\forall (m \leq n) \in \mathbb{N}$: μ_n is a quasi-compact, closed regular Borel probability measure, and $f_{mn} : X_n \to X_m$ is a surjective (onto) quasi-compact–closed continuous function. Then

$$((X, \tau), B(X, \tau), \mu) \triangleq \varprojlim(((X_n, \tau_n), B(X_n, \tau_n), \mu_n), \mathbb{N}, f_{mn})$$

exists and is unique, furthermore, μ is a quasi-compact, closed regular Borel probability measure.

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