Finitely Repeated Games with Monitoring Options*

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Abstract

We study a model of finitely repeated games where the players can decide whether to monitor the other players' actions or not each period. The standard model of repeated games can be interpreted as a model where the players automatically monitor each other. Monitoring is assumed to be private and costless. Hence it is weakly dominant to monitor the other players each period. We thus ask whether the option not to monitor the other players expands the equilibrium payoff vector set. In the context of finitely repeated games with a unique stage game equilibrium, we provide a sufficient condition for a folk theorem when the horizon is sufficiently long.

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1 Introduction

A standard assumption in the models of repeated games is that of perfect monitoring. It assumes that each player directly observes the actions of the others each period. Early folk theorems for discounted infinitely repeated games (Fudenberg and Maskin, 1986) and for finitely repeated games (Benoît and Krishna, 1985) assume perfect monitoring. In some applications, however, this assumption is too strong. This criticism motivated a large body of literature on the repeated games with imperfect monitoring, where each player only receives partial information of the other players' actions.

In this paper, we scrutinize the assumption of perfect observability from a different viewpoint. Namely, the complete information about the past play should come as a consequence of the players' conscious efforts. It needs attention to monitor somebody. Then each player can, in principle, choose not to make the efforts. Therefore, the acquisition of information is part of his decision making, and an incentive problem arises. Some papers address this type of problem, but they all assume that the information acquisition entails costs (Ben-Porath and Kahneman, 2003; Kandori and Obara, 2004; Miyagawa, Miyahara, and Sekiguchi, 2008). In contrast, we assume that the information acquisition is costless. That is, complete data about the actions of the others are freely available each period, and each player simply decides whether to get them or not.

We also assume that each player's information acquisition is completely unobservable to any other player. The other players neither directly observe the decision nor receive any signal of it. As a result, the information acquisition just enables a better decision making, without causing any punishment in itself. Therefore it is always weakly dominant to monitor the other players. This means that any (subgame perfect) equilibrium payoff vector in the standard model, which assumes that the monitoring is automatically made, is an (sequential) equilibrium payoff vector in our model with monitoring options. Consequently, a relevant question to ask is whether an option not to observe the other players can expand the equilibrium payoff vector set. In order to address this question distinctly, we restrict our attention to finitely repeated games with a unique stage-game equilibrium.

It is well known that any finitely repeated game with a unique stage-game Nash equilibrium has a unique subgame perfect equilibrium, which is a repetition of the stage Nash equilibrium. However, in the model with monitoring options, it is possible that a finitely repeated game with a unique stage Nash equilibrium admits a nontrivial sequential equilibrium. Moreover, it is possible that a folk theorem (like Benoît and Krishna, 1985; Smith, 1995; Gossner, 1995) obtains if the players' horizon is sufficiently long. We describe this possibility by first offering a sufficient condition for existence of a sequential equilibrium whose first-period action profile is different from the stage Nash equilibrium, when the times of repetition are long enough. This nontrivial equilibrium, however, does not directly lead to a folk theorem, because its payoff vector may happen to be the same as the one of the trivial equilibrium. Therefore, we offer an additional condition for a folk theorem for players with a sufficiently long horizon.

Our nontrivial equilibria and folk theorem utilize the players' options not to monitor other players, by strategies where a potential deviator is monitored only by a subset of the players. Thus only those monitors detect his deviation, and the other players do not notice it.[1] Then a deviator can be punished by nonequilibrium strategies, where only the past monitors punish him. Since our argument depends on the logic, it is natural that our conditions for nontrivial equilibria and a folk theorem are related to the payoff structure of the *reduced games*, where a subset of players play the stage game under the premise that the players outside the subset play according to the stage game Nash equilibrium. Our conditions basically require a reduced game to satisfy a sufficient condition for a standard folk theorem.

The ability not to monitor other players is related to a player's ability not to respond to past observations. In fact, some forms of commitments are a powerful tool to achieve cooperation in repeated game settings. For example, Renou (2009) studies a model of finite repeated games where at the beginning of play, the players can commit to a subset of their strategy sets. Renou (2009) shows that this type of commitments allows cooperation in various stage games, which include, most strikingly, prisoners' dilemma. However, the result depends on the players' ability to commit not to play certain strategies in the whole course of play. In order to sustain cooperation, the players' horizon must be long. The longer the horizon is, the harder is to make credible commitments.

In this paper, we show how monitoring options allow different equilibria supported by a credible punishment. If we allow non-credible punishments, the monitoring options do not add much to what the players can sustain under the standard model. Therefore, our setting does not improve existing Nash folk theorems by Benoît and Krishna (1987) and González-Díaz (2006).

The rest of this paper is organized as follows. In section 2, we introduce the model and provide our assumptions. In Section 3, we present sufficient conditions for existence of a nontrivial equilibrium and a folk theorem, respectively. In Section 4, we conclude.

2 Model

In this section, we first describe the stage game of the model and then formulate the finitely repeated games. This section also introduces the solution concept, and provides a comparison with the standard model.

2.1 The Stage Game

A finite normal-form game G with n players is given. Let A_i be a finite set of player i's actions, and let $A = \prod_{i=1}^{n} A_i$ be the set of action profiles. $u_i : A \to \mathbb{R}$ is player i's stage payoff function. Let $\mathcal{A}_i$ be the set of probability distributions on A_i, so that $\mathcal{A}_i$ is the set of player i's mixed actions. Let $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$ be the set of mixed action profiles. We extend the domain of each u_i to $\mathcal{A}$, in an obvious way. For each i, let $A_{-i} = \prod_{j \neq i} A_j$ and $\mathcal{A}_{-i} = \prod_{j \neq i} \mathcal{A}_j$. We normalize the players' stage payoffs so that

[1] In order for this to be true, we assume that all players receive stage payoffs, which typically provide some information about actions, only at the end of the whole repeated game in total.

their minimax values are all equal to zero. That is,

$$\min_{\alpha_{-i}\in\mathcal{A}_{-i}} \max_{a_i\in A_i} u_i(a_i, \alpha_{-i}) = 0 \tag{1}$$

for any i.[2]

The stage game of our model consists of playing this game G and then deciding whether to monitor the other players' (realized) actions or not. We assume that the monitoring decision is binary; either to monitor *all* other players' actions or not to monitor any player at all. Namely, we rule out possibilities to only observe a part of the other players. Therefore, we regard each player i's monitoring decision as a choice from a doubleton $M_i = \{0, 1\}$, where 1 denotes observing all other players and 0 denotes monitoring no other player. Unlike the literature which regards monitoring as a consequence of players' costly efforts (Ben-Porath and Kahneman, 2003; Kandori and Obara, 2004; Miyagawa, Miyahara, and Sekiguchi, 2008), we assume that the monitoring decision entails no cost.

Each player i first decides which action $a_i \in A_i$ to choose, and then, depending on his actual choice a_i, chooses $m_i \in M_i$. We allow randomized decisions. Thus a *stage action* of player i is defined as a pair $s_i = (\alpha_i, \mu_i)$, where $\alpha_i \in \mathcal{A}_i$ and μ_i is a function from A_i to $[0, 1]$. For $a_i \in A_i$, $\mu_i(a_i)$ is the probability of observing the other players (in other words, the probability of choosing $m_i = 1$) given that player i has selected a_i. Let S_i be the set of all stage actions. We assume that m_i is completely unobservable to any player $j \neq i$, and a_i is completely unobservable to any other player $j \neq i$ if he chooses $m_j = 0$. This in particular implies that (i) any other information about (a_i, m_i) is available to any player $j \neq i$, and (ii) the players do not receive stage payoffs by the very end of the game. The latter implication must follow, because the stage payoff usually gives information about the other players' actions.[3]

2.2 The Finitely Repeated Game

We consider a finitely repeated game where the stage game described in the previous subsection is played in periods $t = 1, 2, \cdots, T$. We call the game a *T-period repeated game*, and denote it by $G(T)$.

In each period $t \geq 1$, each player i chooses a stage action from S_i, based on his information at the beginning of period t. Player i's information at the beginning of period t consists of all his past actions, all his past monitoring decisions, and the other players' past actions in all periods he decided to monitor them. We assume that current actions can be observed only in the same period, and cannot be observed in a later period. We also assume that the players receive all stage payoffs at the end of period T in total. Thus they cannot learn actions from payoffs.

Let us define each player's strategy more formally. For that purpose, we need to

[2] Unlike early folk theorem literature such as Benoît and Krishna (1985) and Smith (1995), our minimax value concept allows the other players to (independently) randomize.

[3] This is so even when the model exhibits private monitoring, and therefore each player's realized payoff is merely an imperfect signal of the actions. In this case, still the realized payoff provides some information, since its probability distribution usually depends on actions.

define each player's information sets. For $t \geq 2$, let us define

$$H^t = \Big(A \times \{0,1\}^n\Big)^{t-1},$$

which is the set of entire paths at the beginning of period t. Its generic element

$$h^t = \Big\{\big(a(\tau), m(\tau)\big)\Big\}_{\tau=1}^{t-1}$$

is such that $a(\tau) \in A$ and $m(\tau) \in M \equiv \prod_{i=1}^n M_i$, and the i-th element of $m(\tau)$ is interpreted as player i's monitoring decision in period τ. Hence each $h^t \in H^t$ is a whole collection of the players' decisions in all past periods.

For $t \geq 2$, we define *player i's information partition at period t*, denoted by $\mathcal{H}_i^t$, as a partition of H^t with the following property: two elements of H^t, denoted by

$$h^{t,1} = \Big\{\big(a^1(\tau), m^1(\tau)\big)\Big\}_{\tau=1}^{t-1}, \quad h^{t,2} = \Big\{\big(a^2(\tau), m^2(\tau)\big)\Big\}_{\tau=1}^{t-1},$$

belong to the same element of $\mathcal{H}_i^t$ if and only if the following three conditions are satisfied:

(i) $m_i^1(\tau) = m_i^2(\tau)$ for any τ,

(ii) $a_i^1(\tau) = a_i^2(\tau)$ for any τ such that $m_i^1(\tau) = m_i^2(\tau) = 0$, and

(iii) $a^1(\tau) = a^2(\tau)$ for any τ such that $m_i^1(\tau) = m_i^2(\tau) = 1$.

Also it is convenient to define $\mathcal{H}_i^1$, so let $\mathcal{H}_i^1$ be an arbitrary singleton. Then the set of *information sets* of player i is

$$\mathcal{H}_i = \bigcup_{t=1}^{T} \mathcal{H}_i^t.$$

A (behavioral) strategy of player i in $G(T)$ is a function σ_i from $\mathcal{H}_i$ to S_i. Let Σ_i be the set of player i's strategies, and let $\Sigma = \prod_{i=1}^n \Sigma_i$ be the set of strategy profiles. Given a strategy profile $\sigma = (\sigma_i)_{i=1}^n \in \Sigma$, one can compute the probability distribution of the action profile in each period. We assume that player i's expected payoff of the profile σ in $G(T)$ is:

$$\frac{1}{T} E\Bigg[\sum_{t=1}^{T} u_i\big(a(t)\big)\Big|\sigma\Bigg].$$

Namely, the players are interested in maximizing the average, undiscounted sum of the stage payoffs. The assumption of no discounting is made only for easing exposition. Extensions of the results to be reported below to the case of little discounting are straightforward.

2.3 The Solution Concept

Our solution concept is sequential equilibrium, although we sometimes consider Nash equilibrium, too, for the sake of comparison. Since $G(T)$ is a finite extensive-form game, the original definition of sequential equilibrium by Kreps and Wilson (1982) directly applies. An *assessment*, a pair of strategy profile and a system of beliefs, is a *sequential equilibrium* if it satisfies consistency and sequential rationality (in the sense of Kreps and Wilson (1982)).

2.4 Comparisons with the Standard Model

It is instructive to compare our model with the standard model of finitely repeated games (Benoît and Krishna, 1985, for instance). In relation to our model, the standard literature assumes that the players automatically observe the other players. That is, while each player i in our model chooses from $M_i = \{0, 1\}$ each period, a player in the standard model always chooses 1 every period. Let us denote this T-period repeated game by $G^1(T)$.

In our model, monitoring decision is costless and private. Thus acquiring information just enables a better decision making in a future, and it does not cause any punishment because it is unobservable. Abusing terminology slightly, we can say that monitoring the other players is weakly dominant. Hence, intuitively, one should expect that our model can do anything the standard model can do. The following result confirms that intuition.

Proposition 1. *Any subgame perfect equilibrium payoff vector of $G^1(T)$ is a sequential equilibrium payoff vector of $G(T)$.*

Proof. See Appendix A. Q.E.D.

Hence our assumption of monitoring options loses no equilibrium possibilities in the standard model with automatic monitoring. Thus a relevant question to ask is whether it increases the equilibrium payoff vector set.

3 Analysis

We start this section with a motivating example. In order to develop the idea of that example more generally, we next introduce some preliminary notations and definitions. We will limit attention to the case where G has a unique Nash equilibrium. Then we present a central result of this paper, which is a sufficient condition for existence of a nontrivial equilibrium of $G(T)$ with a sufficient large T.

3.1 A Motivating Example

This subsection assumes that G has $n = 3$, $A_i = \{C, D\}$ for any i, and the payoff matrices as follows.

	C	D
C	$10, 10, 10$	$0, 11, 0$
D	$11, 0, 0$	$1, 1, 0$

C

	C	D
C	$1, 1, 1$	$2, 0, 1$
D	$0, 2, 1$	$3, 3, 1$

D

We verify that (D, D, D) is a unique Nash equilibrium of G. To see that, we first claim that under any Nash equilibrium players 1 and 2 choose D with probability 1. Note that C is a best response of player 1 only if player 2 chooses C with a probability no less than 1/2. The same argument holds for player 2. Therefore, if a Nash equilibrium exists such that either player 1 or 2 plays C with a nonzero probability, the equilibrium actually prescribes both players 1 and 2 to choose C with a probability no less than 1/2. Consequently, the action profile $(a_1, a_2) = (C, C)$ must be played with a probability no less than 1/4. But if they play such a profile, C is a unique best response of player 3. Given $a_3 = C$, C is not a best response for either player 1 or 2, a contradiction. As a result, any Nash equilibrium of G must have $a_1 = a_2 = D$. Now it is easy to conclude that (D, D, D) is a unique Nash equilibrium.

Let $T = 2$, so that this game is played just twice. In the standard model, a repetition of (D, D, D) is the only subgame perfect equilibrium outcome. We claim, however, that $G(2)$ has a sequential equilibrium where (C, C, C) is played in the first period. Let us consider the following strategy profile.

Player i's strategy with $i \in \{1, 2\}$. In period 1, player i chooses C, and then chooses $m_i = 1$ irrespective of his action. In period 2,

(i) if he observed the other players in period 1 and if the actions of players 1 and 2 in period 1 was (C, C), then he plays D,

(ii) if he observed the other players in period 1 and if the actions of players 1 and 2 in period 1 was not (C, C), then he plays C,

(iii) if he did not observe the other players in period 1, then he chooses an action he did not choose in period 1, and

(iv) in all those cases, he observes the other players irrespective his action.

Player 3's strategy. In period 1, player 3 chooses C, and then chooses $m_3 = 0$ irrespective of his action. In period 2,

(i) if he did not observe the other players in period 1, then he chooses D,

(ii) if he observed the other players in period 1 and if the actions of players 1 and 2 in period 1 was (C, C), then he plays D,

(iii) if he observed the other players in period 1 and if the actions of players 1 and 2 in period 1 was not (C, C), then he plays C, and

(iv) in all those cases, he observes the other players irrespective his action.

Note that the play under this strategy profile is (C, C, C) in period 1, and (D, D, D) in period 2. We now show that this strategy profile is a sequential equilibrium of $G(2)$, combined with the following system of beliefs.

System of beliefs. At any information set of player i at period 2, whether it is on the path or not, player i believes that the other players did not deviate in his monitoring decision in period 1. Namely, players 1 and 2 are always believed to have monitored the other players, and player 3 is always believed to have observed no other player. This system of beliefs can be made consistent, if we consider trembles such that the probabilities of deviations in terms of monitoring are much smaller than the ones of trembles with respect to actions.

Now we check sequential rationality of this assessment for each player.

Sequential rationality for player i with $i \in \{1, 2\}$. We first check sequential rationality of player i's play in period 2. At any information set at period 2, player i believes that player 3 did not observe the other players, and that player $3 - i$ observed the other players. Hence player i believes that player 3 chooses D in period 2. His belief about the action of player $3 - i$ depends on his observation (if any) in period 1.

- If player i observed the other players in period 1 and if the actions of players 1 and 2 in period 1 was (C, C), then he believes that player $3-i$ plays D in period 2. Hence playing D is sequentially rational.
- If player i observed the other players in period 1 and if the actions of players 1 and 2 in period 1 was not (C, C), then he believes that player $3 - i$ plays C in period 2. Hence playing C is sequentially rational.
- If he did not observe the other players in period 1, then he believes that no other player deviated and that player $3 - i$ plays what player i did not play in period 1. Hence playing what he did not play in period 1 is sequentially rational.
- In all those cases, monitoring does not affect payoffs and is weakly sequentially rational.

Let us now consider the play in period 1. If player i does not deviate, he obtains the payoff of

$$\frac{1}{2}u_i(C, C, C) + \frac{1}{2}u_i(D, D, D) = \frac{13}{2}.$$

If he deviates and plays D, then in the next period player 3 chooses D and player $3-i$ plays C. As we have seen, a sequentially rational action in period 2 is C. Hence the payoff is 11 in period 1 and 1 in period 2, and the average is 6. Hence this deviation does not pay. Failing to observe the other players does not affect the payoff given the subsequent, sequentially rational play. Hence player i's strategy is sequentially rational.

Sequential rationality for player 3. Again we work backwards. At any information set at period 2, player 3 believes that the other players observed the other players. His sequentially rational stage-action is D and monitoring, unless he believes that the other players play (C, C) with a positive probability. This occurs only when player 3 deviantly observed the other players and found that their action pair was not (C, C). Only at that information set, his sequentially rational stage-action is C and monitoring. Since his play in period 2 is exactly as such, his continuation strategy at any information set at period 2 is sequentially rational.

Let us now consider the play in period 1. Note that the action profile in period 1 is (C, C, C), where he plays a static best response. Since the strategies of players 1 and 2 do not depend on player 3's action, playing C and then monitoring no other player are clearly sequentially rational.

Hence we have a sequential equilibrium where mutual cooperation is sustained in period 1. In what follows, we develop the idea of this equilibrium more generally, and seek a sufficient condition for a folk theorem.

3.2 Preliminaries

Let $N = \{1, 2, \cdots, n\}$. Let $\mathcal{N}$ be the set of all proper, nonempty subsets of N. Namely,

$$\mathcal{N} = \{N' \subseteq N : N' \neq \emptyset,\ N' \neq N\}.$$

For $N' \in \mathcal{N}$, define $A_{N'} = \prod_{i \in N'} A_i$ and $\mathcal{A}_{N'} = \prod_{i \in N'} \mathcal{A}_i$. For $N' \in \mathcal{N}$, $a \in A$ and $\alpha \in \mathcal{A}$, let $a_{N'} = (a_i)_{i \in N'}$ and $\alpha_{N'} = (\alpha_i)_{i \in N'}$.

In what follows, we assume:

Assumption 1. G has a unique Nash equilibrium, which we denote by $\alpha^* \in \mathcal{A}$.

Next, we introduce the notion of *reduced games*. Let $N' \in \mathcal{N}$. We define $G_{N'}$ as a normal-form game such that

(i) the set of players is N',

(ii) the action set of each player $i \in N'$ is A_i, and

(iii) the payoff for player $i \in N'$ of an action profile $a_{N'} = (a_i)_{i \in N'}$ is

$$\tilde{u}_i(a_{N'}) = u_i(a_{N'}, \alpha^*_{N \setminus N'}).$$

In other words, $G_{N'}$ is a game where the players in N' play G, on the premise that the players in $N \setminus N'$ follow the unique Nash equilibrium of G, α^*. Note that for any $N' \in \mathcal{N}$, $\alpha^*_{N'}$ is a Nash equilibrium of $G_{N'}$.

3.3 Nontrivial Equilibria

Due to Assumption 1, it is always a sequential equilibrium of $G(T)$ for the players to follow α^* at any information set. We are obviously interested in existence of other sequential equilibria. In this subsection, we seek a sufficient condition for existence of a sequential equilibrium whose initial period (mixed) action profile is different from α^*. The following definition is helpful.

Definition. $N' \in \mathcal{N}$ satisfies *the folk theorem condition* if the corresponding reduced game $G_{N'}$ has the following two properties:

(A) the set of feasible payoff vectors of $G_{N'}$ has a dimension of $|N'|$, and for each $i \in N'$, there exist two Nash equilibria of $G_{N'}$, $\alpha^1_{N'}$ and $\alpha^2_{N'}$, such that

$$\tilde{u}_i(\alpha^1_{N'}) \neq \tilde{u}_i(\alpha^2_{N'}),$$

and

(B) for each $i \in N'$,

$$\tilde{u}_i(\alpha^*_{N'}) > \min_{\alpha_{N'\setminus\{i\}} \in \mathcal{A}_{N'\setminus\{i\}}} \max_{a_i \in A_i} \tilde{u}_i(a_i, \alpha_{N'\setminus\{i\}}). \tag{2}$$

The folk theorem condition consists of two properties. The property (A) is a standard sufficient condition for the folk theorem in finitely repeated games: a pure strategy version by Benoît and Krishna (1985), and its extension to mixed strategies by Gossner (1995). The property (B) is somewhat new, stating that a "default" Nash equilibrium of $G_{N'}$, $\alpha^*_{N'}$, gives each player more than his minimax value. Those two properties guarantee that all finitely repeated games with a sufficiently long horizon and with a stage game $G_{N'}$ have for each player an equilibrium giving him a payoff smaller than the payoff of the default Nash equilibrium $\alpha^*_{N'}$.

We point out that the folk theorem condition cannot hold under any *two-player* game. With only two players, $G_{N'}$ is always a one-person game. Such games always have a unique equilibrium payoff, and therefore the property (A) always fails.

Now we are ready to prove a result for existence of a nontrivial equilibrium.

Proposition 2. *Suppose some $N' \in \mathcal{N}$ satisfies the folk theorem condition. Then there exists $\underline{T}$ such that any $G(T)$ with $T \geq \underline{T}$ has a sequential equilibrium whose action profile in period 1 is not α^*.*

Proof. See Appendix B. Q.E.D.

It is important to notice that the result does not have implications on equilibrium *payoffs*. It is possible that the nontrivial equilibrium happens to have the same payoff vector as α^*. If that is the case, we have a nontrivial equilibrium but do not have a nontrivial equilibrium payoff vector. Therefore, in order to prove a folk theorem result based on Proposition 2, we need to make another assumption on the payoff structure of G. This is a subject of the next subsection.[4]

3.4 A Folk Theorem

This subsection establishes a folk theorem for finitely repeated games with monitoring options.

Definition. $N' \in \mathcal{N}$ satisfies *the feasibility condition for player i* if there exist a pure action profile $a_{N'} \in A_{N'}$ and a (possibly mixed) action profile $\alpha_{N\setminus N'} \in \mathcal{A}_{N\setminus N'}$ such that

[4] However, we feel that the additional condition to be presented in the next subsection is redundant, which is actually implied by what we have already assumed. In fact, if the condition of Proposition 2 is satisfied, then for any action profile $a_{N'} \in A_{N'}$, a sequential equilibrium exists where the players in N' play $a_{N'}$ in the initial period. Therefore, we have flexibilities on the choice of actions in the first period. A folk theorem would obtain unless all of them give one player a common stage-payoffs. But we are unable to prove it formally.

(I) for any $j \notin N'$,

$$u_j\big(a_{N'}, \alpha_{N\setminus N'}\big) = \max_{a_j \in A_j} u_j\big(a_j, (a_{N'}, \alpha_{N\setminus N'})_{-j}\big),$$

and

(II) $u_i\big(a_{N'}, \alpha_{N\setminus N'}\big) \neq u_i(\alpha^*)$.

Remark 1. In the above definition, it does not matter whether $i \in N'$ or not.

$N' \in \mathcal{N}$ satisfies the feasibility condition for player i if there exists a mixed action profile such that (i) any player in N' chooses a pure action, (ii) any player outside N' plays a best response, and (iii) player i receives a different stage-payoff than α^*.

With this terminology, together with the one we introduced in the previous subsection, we have a sufficient condition for a folk theorem. Let F be a convex hull of the set $\big\{(u_i(a))_{i=1}^n : a \in A\big\}$. Namely, F is the set of feasible payoff vectors. Since we have normalized each player's minimax payoff to zero, the set of feasible and (weakly) individually rational payoff vectors is $F^* \equiv F \cap \mathbb{R}_+^n$.

Proposition 3. *Assume that for any i, there exists $N' \in \mathcal{N}$ which satisfies the folk theorem condition and the feasibility condition for player i. Also assume that the dimension of F is n. Then for any $\varepsilon > 0$, there exists $\underline{T}$ such that for any $T \geq \underline{T}$ and any $v \in F^*$, there exists a sequential equilibrium of $G(T)$ whose payoff vector $(w_i)_{i=1}^n$ satisfies $|v_i - w_i| < \varepsilon$ for any $i \in N$.*

Proof. Fix i and $N' \in \mathcal{N}$ which satisfies the folk theorem condition and the feasibility condition for player i. Let $\hat{\alpha} = \big(a_{N'}, \alpha_{N\setminus N'}\big)$ be a mixed action profile satisfying the properties (I) and (II). By Proposition 2 and its proof, a natural number $\underline{T}_i$ exists such that any $G(T)$ with $T \geq \underline{T}_i$ has a sequential equilibrium with a path such that $\hat{\alpha}$ is played in the first period and then α^* is played in all subsequent periods. By the property (II), player i's payoff of this equilibrium is not $u_i(\alpha^*)$. Let $\underline{T}_0 = \max_i \underline{T}_i$. Then any $G(T)$ with $T \geq \underline{T}_0$ has multiple sequential equilibrium payoffs for each player. Depending on this multiplicity and resorting to the folk theorem by Gossner (1995), we can prove that the statement is true if we choose $\underline{T}$ large enough. Q.E.D.

3.5 Discussions

In this subsection, we discuss our conditions by way of examples.

Example 1. Let us reproduce the game we studied in Subsection 3.1:

C

	C	D
C	10, 10, 10	0, 11, 0
D	11, 0, 0	1, 1, 0

D

	C	D
C	1, 1, 1	2, 0, 1
D	0, 2, 1	3, 3, 1

This game satisfies the conditions for Propositions 2 and 3. If we let $N' = \{1, 2\}$, the reduced game $G_{\{1,2\}}$ has two Nash equilibrium payoff vectors $(1, 1)$ and $(3, 3)$. The minimax value of each player in $G_{\{1,2\}}$ is 1, which is greater than 3, the payoff of the default Nash equilibrium (D, D). Therefore, $\{1, 2\}$ satisfies the folk theorem condition. Since $u_i(C, C, C) = 10$ for any i and $a_3 = C$ is a best response against $(a_1, a_2) = (C, C)$, $\{1, 2\}$ also satisfies the feasibility condition for any i.

Example 2. Again, suppose G has $n = 3$, $A_i = \{C, D\}$ for any i. Now the payoff matrices are as follows.

	C	D
C	10, 10, 10	0, 11, 0
D	11, 0, 0	1, 1, 0

C

	C	D
C	3, 3, 1	0, 2, 1
D	2, 0, 1	1, 1, 1

D

The same line of argument as in Example 1 proves that (D, D, D) is a unique Nash equilibrium of G. If we set $N' = \{1, 2\}$, the reduced game $G_{\{1,2\}}$ again has two Nash equilibrium payoff vectors $(1, 1)$ and $(3, 3)$. However, now each player's minimax value equals 1, which is the payoff of the default equilibrium (D, D). Hence $\{1, 2\}$ does not satisfy the folk theorem condition. Any other $N' \in \mathcal{N}$ does not satisfy it. For example, if $N' = \{1, 3\}$, the reduced game $G_{\{1,3\}}$ has a unique equilibrium. Hence the folk theorem condition does not hold for $N' = \{1, 3\}$, and similarly for $N' = \{2, 3\}$. We do now know whether a finitely repeated game with this stage game has a nontrivial equilibrium or not.

4 Concluding Remarks

Our sufficient conditions for existence of a nontrivial equilibrium and for a folk theorem do not seem to be necessary. Moreover, our conditions do not apply to any two-player model. More work on the two-player games will facilitate our understanding of the model with monitoring options. We are now working toward that direction.

A Appendix: Proof of Proposition 1

First of all, we define for each player i his set of information sets where he has selected $m_i = 1$ in all previous periods. We denote this set by $\hat{\mathcal{H}}_i$. Formally, $\hat{\mathcal{H}}_i$ is defined as

$$\hat{\mathcal{H}}_i = \bigcup_{t=1}^{T} \hat{\mathcal{H}}_i^t,$$

where each $\hat{\mathcal{H}}_i^t$ is defined as:

- $\hat{\mathcal{H}}_i^1 = \mathcal{H}_i^1$, and
- for $t \geq 2$, $\hat{\mathcal{H}}_i^t$ is a subset of $\mathcal{H}_i^t$ such that $\iota_i^t \in \mathcal{H}_i^t$ belongs to $\hat{\mathcal{H}}_i^t$ if and only if any element of ι_i^t, denoted $h^t = \{(a(\tau), m(\tau))\}_{\tau=1}^{t-1}$, satisfies $m_i(\tau) = 1$ for any τ.

Let $\sigma^1 = (\sigma_i^1)_{i=1}^n$ be a subgame perfect equilibrium of $G^1(T)$. Then for each i, each σ_i^1 can be seen as a function from $\hat{\mathcal{H}}_i$ to $\mathcal{A}_i$. Now let us define a strategy of player i in $G(T)$, denoted by $\hat{\sigma}_i$, as follows.

- for any $\iota_i \in \hat{\mathcal{H}}_i$, let

$$\hat{\sigma}_i(\iota_i) = \big(\sigma_i^1(\iota_i), \bar{\mu}_i^1\big),$$

  where $\bar{\mu}_i^1$ is a function on A_i such that $\bar{\mu}_i^1(a_i) = 1$ for any $a_i \in A_i$, and

- for any information set in $\mathcal{H}_i \setminus \hat{\mathcal{H}}_i$, assign an arbitrary stage action.

Namely, $\hat{\sigma}_i$ is a strategy where player i plays G in each period in the same way as σ_i^1 and then always monitors the other players, on any information set where he has observed the other players in all past periods. His behavior at histories where he has not observed them in some period is arbitrary. Let $\hat{\sigma} = (\hat{\sigma}_i)_{i=1}^n$.

Let Ψ^* be a system of beliefs such that

- the assessment $(\hat{\sigma}, \Psi^*)$ is consistent, and
- any player i at any history believes that the other players have monitored all players in all past periods.

The latter requirement can be satisfied, by considering trembles that put far less weights on the deviations in monitoring than the ones in actions.

For each i, let $\sigma_i^* \in \Sigma_i$ be such that

- σ_i^* coincides with $\hat{\sigma}_i$ on $\hat{\mathcal{H}}_i$, and
- for any information in $\mathcal{H}_i \setminus \hat{\mathcal{H}}_i$, the continuation strategy of σ_i^* at that information set is sequentially rational given $(\hat{\sigma}_{-i}, \Psi^*)$.

Let $\sigma^* = (\sigma_i^*)_{i=1}^n$.

Let us examine whether the assessment (σ^*, Ψ^*) is a sequential equilibrium of $G(T)$. First, the same trembles which make Ψ^* consistent under $(\hat{\sigma}, \Psi^*)$ also make it consistent under (σ^*, Ψ^*). Second, let us examine sequential rationality. At any information set of player i in $\mathcal{H}_i \setminus \hat{\mathcal{H}}_i$, his belief about the other players' continuation strategies is the same as the one under $(\hat{\sigma}, \Psi^*)$. This is because he believes that any other player $j \neq i$ is at an information in $\hat{\mathcal{H}}_j$, and $\hat{\sigma}_j$ and σ_j^* coincide on $\hat{\mathcal{H}}_j$. Therefore, his continuation strategy at the information set is sequentially rational by the definition of σ^*. Next, consider an information set in $\hat{\mathcal{H}}_i$. Since monitoring is costless and completely unobservable to the other players, there exists a sequentially rational continuation strategy which prescribes monitoring in all subsequent periods. Since σ^1 is a subgame perfect equilibrium of $G^1(T)$, the continuation strategy of σ_i^* is sequentially rational. Hence the assessment is sequentially rational. Since its path of action profiles is the same as that of σ, it achieves the same payoff vector as σ.

B Appendix: Proof of Proposition 2

Fix $N' \in \mathcal{N}$ satisfying the folk theorem condition. Since the property (B) holds for any $i \in N'$, $\varepsilon > 0$ exists such that

$$\tilde{u}_i\big(\alpha_{N'}^*\big) - \varepsilon > \min_{\alpha_{N' \setminus \{i\}} \in \mathcal{A}_{N' \setminus \{i\}}} \max_{a_i \in A_i} \tilde{u}_i\big(a_i, \alpha_{N' \setminus \{i\}}\big) \tag{3}$$

for any $i \in N'$.

Fix $a_{N'} \in A_{N'}$ such that $a_{N'} \neq \alpha^*_{N'}$. Then $\alpha_{N \setminus N'} \in \mathcal{A}_{N \setminus N'}$ exists such that, when we write $\hat{\alpha} = \big(a_{N'}, \alpha_{N \setminus N'}\big)$,

$$u_j(\hat{\alpha}) = \max_{a_j \in A_j} u_j\big(a_j, \hat{\alpha}_{-j}\big) \tag{4}$$

for any $j \notin N'$. Since $a_{N'} \neq \alpha^*_{N'}$, $\hat{\alpha}$ is not a Nash equilibrium of G by Assumption 1. Therefore, (4) implies that

$$\Delta \equiv \max_{i \in N'} \Big[\max_{a_i \in A_i} u_i\big(a_i, \hat{\alpha}_{-i}\big) - u_i\big(\hat{\alpha}\big) \Big] > 0.$$

Let us choose a natural number T_0 so that

$$\Delta < T_0 \varepsilon. \tag{5}$$

For any T', let $G^1_{N'}(T')$ be a T'-period repeated game with an automatic monitoring whose stage game is $G_{N'}$. By the property (A), the folk theorem by Gossner (1995) applies. By (3), a natural number T_1 exists such that for any $T' \geq T_1$ and any $i \in N'$, a subgame perfect equilibrium of $G^1_{N'}(T')$, $\sigma^{1,i}$, exists such that its payoff for player i is smaller than $u_i(\alpha^*) - \varepsilon$.

Let $\underline{T} = \max\{T_0, T_1\} + 1$. Fix $T \geq \underline{T}$, and let $G_{N'}(T-1)$ be a $(T-1)$-period repeated game with monitoring options whose stage game is $G_{N'}$. Since $T - 1 \geq T_1$, by Proposition 1 for each player i in N' there exists a sequential equilibrium of $G_{N'}(T-1)$, denoted by $\tilde{\sigma}^i$, whose payoff for player i is smaller than $u_i(\alpha^*) - \varepsilon$.

For $i \in N'$ and $j \in N'$, we define a strategy of $G(T-1)$, denoted by $\hat{\sigma}^j_i$, as an extension of $\tilde{\sigma}^j_i$ (it is a strategy of $G_{N'}(T-1)$) in a natural way. Namely, $\hat{\sigma}^j_i$ is such that (i) it does not depend on any observations (if any) of the players in $N \setminus N'$, and (ii) it depends on his own past actions and his observations about the players in N' (if any) in the same way as $\tilde{\sigma}^j_i$.

For $i = 1, 2, \cdots, n$, let $\bar{\mu}^z_i$ ($z \in \{0, 1\}$) be a function on A_i such that $\bar{\mu}^z_i(a_i) = z$ for any $a_i \in A_i$. Now we define another strategy of $G(T-1)$, denoted by $\hat{\sigma}^0_i$, for each $i = 1, 2, \cdots n$. For $i \in N'$, $\hat{\sigma}^0_i$ is a strategy which specifies a stage action $s_i = \big(\alpha^*_i, \bar{\mu}^1_i\big)$ at any information set. For $i \notin N'$, $\hat{\sigma}^0_i$ is a strategy which specifies a stage action $s_i = \big(\alpha^*_i, \bar{\mu}^0_i\big)$ at any information set.

Now we are ready to construct an assessment of $G(T)$, whose strategy profile is denoted by $\sigma^* = \big(\sigma^*_i\big)^n_{i=1}$, and system of beliefs by Ψ^*. We first specify each σ^*_i.

Stage actions in period 1. Each player $i \in N'$ chooses $s_i = \big(\hat{\alpha}_i, \bar{\mu}^1_i\big)$, and each player $i \notin N'$ chooses $s_i = \big(\hat{\alpha}_i, \bar{\mu}^0_i\big)$.

Continuation strategies at information sets at period 2. We first consider each player $i \notin N'$.

(A) If player i chose $m_i = 0$ in period 1, then his continuation strategy is $\hat{\sigma}^0_i$.

(B) If player i chose $m_i = 1$ in period 1, let $a(1) \in A$ be the combination of his own action and his observations. Now player i's continuation strategy is a strategy which is sequentially rational, given Ψ^* (to be specified below) and the following strategy profile of the other players:

- player $j \in N'$ plays $\hat{\sigma}_j^k$ if $k \in N'$ exists such that $a_k(1) \neq \hat{\alpha}_k$ and $a_j(1) = \hat{\alpha}_j$ for any $j \in N' \setminus \{k\}$ (recall that $\hat{\alpha}_j$ is pure for any $j \in N'$), and player $j \in N'$ plays $\hat{\sigma}_j^0$ if no such k exists (this includes the case of $a_{N'}(1) = \hat{\alpha}_{N'}$ and the cases where two or more players in N' play differently from $\hat{\alpha}$), and
- player $j \notin N'$ plays $\hat{\sigma}_j^0$.

Next we consider each player $i \in N'$.

(a) If player i chose $m_i = 1$ in period 1, let $a(1) \in A$ be the combination of his own action and his observations. If $k \in N'$ exists (it may be that $k = i$) such that $a_k(1) \neq \hat{\alpha}_k$ and $a_j(1) = \hat{\alpha}_j$ for any $j \in N' \setminus \{k\}$, player i's continuation strategy is $\hat{\sigma}_i^k$. If no such k exists, player i's continuation strategy is $\hat{\sigma}_i^0$.

(b) If player i chose $m_i = 0$ in period 1, let $a_i(1)$ be his own action in period 1. Player i's continuation strategy is $\hat{\sigma}_i^0$ if $a_i(1) = \hat{\alpha}_i$, and $\hat{\sigma}_i^i$ if $a_i(1) \neq \hat{\alpha}_i$.

We turn to the system of beliefs Ψ^*. It is specified so that the assessment is consistent, and it satisfies the following additional properties.

- at any information of some player, he believes that the other players did not deviate in their monitoring decisions, and
- furthermore, at any information set of some player where he did not observe the other players in period 1, he believes that the other players did not deviate in terms of actions in period 1.

Let us examine whether the assessment (σ^*, Ψ^*) is a sequential equilibrium of $G(T)$. First, one can see that it is made consistent by trembles putting far less weights on the deviations in terms of monitoring decisions in every period *and* in terms of actions in period 1, than the deviations in terms of actions from period 2 on. Second, let us examine sequential rationality.

Sequential rationality for player $i \notin N'$. We first check sequential rationality of the continuation strategies from period 2 on.

(A) If player i chose $m_i = 0$ in period 1, then player i believes that no other player deviated in period 1. Hence player i believes that any other player $j \neq i$ plays $\hat{\sigma}_j^0$ from period 2 on. Therefore, $\hat{\sigma}_i^0$ is sequentially rational. In fact, player i himself may deviate and reach to an information set at a subsequent period where he learns that some other player did not conform to $\hat{\sigma}_{-i}^0$. However, under Ψ^*, he still believes that no other player has deviated in period 1. Therefore he continues to believe that the other players conform to $\hat{\sigma}_{-i}^0$ in all subsequent periods, and then conforming to $\hat{\sigma}_i^0$ is again sequentially rational.

(B) If player i chose $m_i = 1$ in period 1, let $a(1) \in A$ be the combination of his own action and his observations. Under Ψ^*, player i believes that no other player deviated in monitoring in period 1. Hence he believes that any player $j \in N'$ plays a continuation strategy prescribed by σ_j^*, given that he played $a_j(1)$ and observed $a_{-j}(1)$ in period 1, and believes that any player $j \notin N'$ plays $\hat{\sigma}_j^0$. Then by definition, his continuation strategy is sequentially rational. Again, he may (deviantly) reach to an information set where his observations are inconsistent with the other players' continuation strategies. Under Ψ^*, he still believes that the other players conform to the profile from the next period on, against which conforming to his own continuation strategy is sequentially rational.

Now we consider the stage action in period 1. In period 1, player i plays his static best response $\hat{\alpha}_i$ (see (4)). Also he expects that no other player j's continuation strategy depends on player i's action, which is $\hat{\sigma}_j^0$. Therefore player i has nothing to learn in period 1. This shows that $s_i = \left(\hat{\alpha}_i, \bar{\mu}_i^0\right)$ is a sequentially rational stage action in period 1.

Sequential rationality for player $i \in N'$. We first check sequential rationality of the continuation strategies from period 2 on.

(A) If player i chose $m_i = 1$ in period 1, let $a(1) \in A$ be the combination of his own action and his observations. Player i believes that no other player deviated in monitoring in period 1. Therefore he believes that any player $j \notin N'$ plays $\hat{\sigma}_j^0$. If $k \in N'$ exists such that $a_k(1) \neq \hat{\alpha}_k$ and $a_j(1) = \hat{\alpha}_j$ for any $j \in N' \setminus \{k\}$, player i believes that any player $j \in N'$ plays $\hat{\sigma}_j^k$, and under Ψ^* he continues to believe so at any subsequent information set. Hence $\hat{\sigma}_i^k$ is sequentially rational. If no such $k \in N'$ exists, player i believes that any player $j \in N'$ plays $\hat{\sigma}_j^0$, and under Ψ^* he continues to believe so at any subsequent information set. Hence $\hat{\sigma}_i^0$ is sequentially rational.

(B) If player i chose $m_i = 0$ in period 1, let $a_i(1)$ be his own action in period 1. Again, player i believes that no other player deviated in monitoring in period 1. Therefore he believes that any player $j \notin N'$ plays $\hat{\sigma}_j^0$. If $a_i(1) \neq \hat{\alpha}_i$, player i believes that any player $j \in N'$ plays $\hat{\sigma}_j^i$, and under Ψ^* he continues to believe so at any subsequent information set. Hence $\hat{\sigma}_i^i$ is sequentially rational. If $a_i(1) = \hat{\alpha}_i$, player i believes that any player $j \in N'$ plays $\hat{\sigma}_j^0$, and under Ψ^* he continues to believe so at any subsequent information set. Hence $\hat{\sigma}_i^0$ is sequentially rational.

Now we consider the stage action in period 1. In period 1, player i follows σ_i^*, then his payoff in $G(T)$ is

$$\frac{1}{T} u_i(\hat{\alpha}) + \frac{T-1}{T} u_i(\alpha^*). \tag{6}$$

If he plays $a_i \neq \hat{\alpha}_i$, then his stage-payoff increases at most by Δ. From the next period on, player $j \notin N'$ plays $\hat{\sigma}_j^0$, and all players in N' (including player i himself) play $\hat{\sigma}^i$. Since the payoff for player i of $\tilde{\sigma}^i$ in $G_{N'}(T-1)$ is less than $u_i(\alpha^*) - \varepsilon$. Hence his payoff when he plays $a_i \neq \hat{\alpha}_i$ in period 1 is at most

$$\frac{1}{T}\Big(u_i(\hat{\alpha}) + \Delta\Big) + \frac{T-1}{T}\Big(u_i(\alpha^*) - \varepsilon\Big). \tag{7}$$

By $T \geq T_0$ and (5), we conclude that (6) is greater than (7). Hence $s_i = \left(\hat{\alpha}_i, \bar{\mu}_i^1\right)$ is a sequentially rational stage action in period 1.

Hence the assessment is sequentially rational, and the proof is complete.

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