Young's axiomatization of the Shapley value - a new proof*

Miklós Pintér
Corvinus University of Budapest†

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Abstract

Young's characterization of the Shapley value is considered. A new proof of this axiomatization is presented, moreover, as applications of the new proof, it is demonstrated that the axioms under consideration characterize the Shapley value on various well-known subclasses of TU games.

1 Introduction

In this paper we look into one of the well-known characterizations of the Shapley value [7], into Young's [8] axiomatization. The Shapley value is probably the most popular one-point solution of transferable utility (TU) cooperative games (henceforth games). It is applied in various fields ranging from medicine to statistics, from engineering to accounting etc. Therefore a solid characterization could well serve, among others, applications by helping in understanding its very nature.

Young axiomatized the Shapley value with three axioms: efficiency (Pareto optimality PO), symmetry or equal treatment property ETP (although symmetry and ETP are different axioms, they are equivalent on the class of TU games), and strong monotonicity. Later Moulin [3] suggested another proof for Young's result for the three player case. Both Young's and Moulin's results are applied to the whole class of TU games, however Young also showed that the given characterization is valid on the class of superadditive games as well.

With respect to other subclasses on which Young's result is valid we have to mention three other papers. Neyman [5] showed that for any game, a value defined on the additive group generated by the given game, is efficient (PO), symmetric (or ETP), and strongly monotonic if and only if it is the Shapley value. Khmelnitskaya [1] provided the same result for the class of non-negative constant-sum games with non-zero worth of grand coalition and for the entire

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†Department of Mathematics, Corvinus University of Budapest, 1093 Hungary, Budapest, Fővám tér 13-15., miklos.pinter@uni-corvinus.hu

class of constant-sum games. Moroever, Mlodak [2] applied the same method as Khmelnitskaya to characterize the Shapley value on the class of non-negative bilateral games.

The main motivation of this paper is mathematical. We provide a new proof for Young's axiomatization of the Shapley value. Our new proof makes it possible to prove the validity of Young's characterization of the Shapley value on some other subclasses of games not considered in the literature yet.

The setup of the paper is as follows. In Section 2., we introduce the terminology used throughout the paper. Section 3. discusses our main result. Moreover, some proofs are relegated to the Appendix.

2 Preliminaries

First, some notation: $|N|$ is for the cardinality of set N, and 2^N denotes the set of all subsets of N. $A \subset B$ means $A \subseteq B$, but $A \neq B$. Moreover, we use $|a|$ as well for the absolute value of a real number $a \in \mathbb{R}$.

Let $N \neq \emptyset$, $|N| < \infty$, and $v : 2^N \to \mathbb{R}$ be such a function that $v(\emptyset) = 0$. Then N, v are called set of players, and *transferable utility cooperative game* (henceforth game) respectively. The class of games with players' set N is denoted by $\mathcal{G}^N$.

Let $v \in \mathcal{G}^N$ and $i \in N$ be arbitrarily fixed, and $\forall S \subseteq N$: $v'_i(S) = v(S \cup \{i\}) - v(S)$. Then v'_i is called players i's *marginal contribution function* in game v. Put it differently, $v'_i(S)$ is player i's marginal contribution to coalition S in game v.

In this paper, along with $\mathcal{G}^N$, we consider also subclasses of games defined below. A game $v \in \mathcal{G}^N$ is

- essential, if $v(N) > \sum\limits_{i \in N} v(\{i\})$,
- convex, if $\forall S, T \subseteq N$: $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$,
- strictly convex, if $\forall S, T \subseteq N$, $S \nsubseteq T$, $T \nsubseteq S$: $v(S) + v(T) < v(S \cup T) + v(S \cap T)$,
- superadditive, if $\forall S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) \leq v(S \cup T)$,
- strictly superadditive, if $\forall S, T \subseteq N$, $S, T \neq \emptyset$, $S \cap T = \emptyset$: $v(S) + v(T) < v(S \cup T)$,
- weakly superadditive, if $\forall S \subseteq N$, $\forall i \in N \setminus S$: $v(S) + v(\{i\}) \leq v(S \cup \{i\})$,
- strictly weakly superadditive, if $\forall S \subseteq N$, $S \neq \emptyset$, $\forall i \in N \setminus S$: $v(S) + v(\{i\}) < v(S \cup \{i\})$,
- monotonic, if $\forall S, T \subseteq N$, $S \subseteq T$: $v(S) \leq v(T)$,
- strictly monotonic, if $\forall S, T \subseteq N$, $S \subset T$: $v(S) < v(T)$,
- additive, if $\forall S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) = v(S \cup T)$,
- weakly subadditive, if $\forall S \subseteq N$, $\forall i \in N \setminus S$: $v(S) + v(\{i\}) \geq v(S \cup \{i\})$,

- strictly weakly subadditive, if $\forall S \subseteq N$, $S \neq \emptyset$, $\forall i \in N \setminus S$: $v(S) + v(\{i\}) > v(S \cup \{i\})$,
- subadditive, if $\forall S, T \subseteq N$, $S \cap T = \emptyset$: $v(S) + v(T) \geq v(S \cup T)$,
- strictly subadditive, if $\forall S, T \subseteq N$, $S, T \neq \emptyset$, $S \cap T = \emptyset$: $v(S) + v(T) > v(S \cup T)$,
- concave, if $\forall S, T \subseteq N$: $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$,
- strictly concave, if $\forall S, T \subseteq N$, $S \not\subseteq T$, $T \not\subseteq S$: $v(S) + v(T) > v(S \cup T) + v(S \cap T)$.

For the definition of an essential game cf. von Neumann and Morgenstern [4], and for other types of games cf. Peleg and Sudhölter [6].

It is well known that an equivalent definition of (strictly) convex / (strictly) concave games might be given as following:

$$v \in \mathcal{G}^N \text{ is a (strictly) convex / (strictly) concave if and only if } \forall i \in N, \forall T, Z \subseteq N \setminus \{i\} \text{ such that } Z \subset T\text{: } v'_i(Z) \leq v'_i(T) \ (v'_i(Z) < v'_i(T)) \ / \ v'_i(Z) \geq v'_i(T) \ (v'_i(Z) > v'_i(T)). \tag{1}$$

The *dual* of a game $v \in \mathcal{G}^N$ is such a game $\bar{v} \in \mathcal{G}^N$ that $\forall S \subseteq N$: $\bar{v}(S) = v(N) - v(N \setminus S)$.

Let $v \in \mathcal{G}^N$ be arbitrarily fixed. $i \sim^v j$ $(i, j \in N)$, if $\forall S \subseteq N$ such that $i, j \notin S$: $v'_i(S) = v'_j(S)$. It is easy to verify that for any game $v \in \mathcal{G}^N$: $\sim^v$ is a binary equivalence relation on $N \times N$.

Furthermore, if $S \subseteq N$ is such that $\forall i, j \in S$: $i \sim^v j$, then we say that S is an *equivalence class* in v.

Next we summarize some important properties of dual games. For arbitrary game $v \in \mathcal{G}^N$:

$$\text{If } i \sim^v j \text{ then } i \sim^{\bar{v}} j. \text{ Furthermore, the dual of a (strictly) convex game is a (strictly) concave game, and the dual of a (strictly) concave game is a (strictly) convex game.} \tag{2}$$

Throughout the paper we consider single-valued solutions. A function $\psi : A \to \mathbb{R}^N$, defined on $A \subseteq \mathcal{G}^N$, is called a *solution* on set of games A.

For arbitrary game $v \in \mathcal{G}^N$ the *Shapley solution* ϕ is given by

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} v'_i(S) \frac{|S|!(|N \setminus S| - 1)!}{|N|!} \quad i \in N,$$

where $\phi_i(v)$ is also called player i's *Shapley value* [7].

The solution ψ on $A \subseteq \mathcal{G}^N$ is / satisfies

- *Pareto optimal* (PO), if $\forall v \in A$: $\sum_{i \in N} \psi_i(v) = v(N)$,
- *equal treatment property* (ETP), if $\forall v \in A, \forall i, j \in N$: $(i \sim^v j) \Rightarrow (\psi_i(v) = \psi_j(v))$,

- *equal marginality property* (EMP), if $\forall v, w \in A$, $\forall i \in N$: $(v'_i = w'_i) \Rightarrow (\psi_i(v) = \psi_i(w))$.

Remark 2.1. It is very important to note that the Shapley solution is (completely) determined by the marginal contribution functions. Therefore we can claim that for any EMP solution ψ: if $\psi_i(v) = \phi_i(v)$ and $v'_i = w'_i$ then $\psi_i(w) = \phi_i(w)$.

It is well known and not difficult to check that the Shapley solution is PO, ETP, and EMP.

3 The main result

In this section we present our main result. The following example illustrates the strategy of the proof of Theorem 3.3.

Example 3.1. Let $N = \{1,2,3\}$ and $v = (0,0,0,3,1,2,3) \in \mathcal{G}^N$, where $v = (v(\{1\}), \ldots, v(\{|N|\}), v(\{1,2\}), \ldots, v(\{|N|-1, |N|\}), \ldots, v(N))$. Then v is a superadditive but not convex game, and $1 \nsim^v 2$, $1 \nsim^v 3$, $2 \nsim^v 3$.

Furthermore, let ψ be a PO, ETP, and EMP solution on $\mathcal{G}^N$. We show that $\psi_2(v) = \phi_2(v)$.

Fix player 1 and chose player 2. Then there is such a game $w = (0,0,0,3,2,2,4)$ that $w'_2 = v'_2$ and $1 \sim^w 2$ (it is clear that, w is not the only game in which players 1 and 2 are equivalent and $w'_2 = v'_2$).

Next take set $\{1,2\}$ and chose player 3. Then there is such a game $z = (0,0,0,2,2,2,3)$ that $z'_3 = w'_3$ and $1 \sim^z 2 \sim^z 3$.

Then PO and ETP imply that $\psi(z) = \phi(z)$. Moreover, by EMP $\psi_3(w) = \phi_3(w)$. Since ψ is PO and ETP, $1 \sim^w 2$, therefore $\psi(w) = \phi(w)$.

By applying EMP again, we get $\psi_2(v) = \phi_2(v)$.

From Example 3.1. it is clear that we can deduce $\psi_i = \phi_i$ for any i (player). In other words, we can show that $\psi(v) = \phi(v)$. All we need is that, ψ must be defined on the paths from v to z (different w and z for different i).

The next notion is an important ingredient of our main theorem.

Definition 3.2. *Set $A \subseteq \mathcal{G}^N$ is EMP-closed, if $\forall v \in A$, $\forall S \subseteq N$ equivalence class in v, and $\forall k \in N \setminus S$: $\exists w \in A$ such that $S \cup \{k\}$ is an equivalence class in w and $w'_k = v'_k$.*

The following theorem is our main result.

Theorem 3.3. *Let set $A \subseteq \mathcal{G}^N$ be such that $\forall v \in A$, $\forall k \in N$, and $\forall i \in N \setminus \{k\}$: $\exists B \subseteq A$, $\exists w \in A$, and $\exists z(i) \in B$ such that*

1. *B is EMP-closed,*
2. *$w'_k = v'_k$ and $\forall i \in N \setminus \{k\}$: $z(i)'_i = w'_i$.*

Then ψ a solution on A is PO, ETP, and EMP if and only if $\psi = \phi$.

The above theorem is not tight. Consider e.g. the subclass of three player games, $N = \{1,2,3\}$, which contains only three games: the zero game, the unanimity game on coalition $\{1,2\}$ $u_{\{1,2\}}$, and $v = (0,0,1,1,1,1,2)$. It is a slight calculation to show that this class does not meet the conditions of Theorem 3.3., but axioms PO, ETP, and EMP characterize the Shapley value on it.

The proof of Theorem 3.3. if : It is well-known.

only if: Let $v \in A$ be arbitrarily fixed, and $n = |N|$. Moreover, let $i_1 \in N$ be arbitrarily fixed and $i_2 \in N \setminus \{i_1\}$ be also arbitrarily fixed. Let B be an EMP-closed class of games as in the head of the theorem, i.e. it depends on v and i_1, furthermore, let $z \in B$ be arbitrarily fixed.

B is EMP-closed therefore $\exists z(1) \in B$ such that $z(1)'_{i_2} = z'_{i_2}$ and $\{i_1, i_2\}$ is an equivalence class in $z(1)$. Let $i_3 \in N \setminus \{i_1, i_2\}$ be arbitrarily fixed.

B is EMP-closed therefore $\exists z(2) \in B$ such that $z(2)'_{i_3} = z(1)'_{i_3}$ and $\{i_1, i_2, i_3\}$ is an equivalence class in $z(2)$. Let $i_4 \in N \setminus \{i_1, i_2, i_3\}$ be arbitrarily fixed.

$\vdots$

B is EMP-closed therefore $\exists z(n-1) \in B$ such that $z(n-1)'_{i_n} = z(n-2)'_{i_n}$ and $\{i_1, i_2, \dots, i_n\}$ $(= N)$ is an equivalence class in $z(n-1)$.

By PO and ETP, $\psi(z(n-1)) = \phi(z(n-1))$ because all players $i_1, i_2, \dots, i_n$ are symmetric in $z(n-1)$.

Since both values ψ and ϕ meet EMP and by construction $z(n-1)'_{i_n} = z(n-2)'_{i_n}$, it follows that $\psi_{i_n}(z(n-2)) = \psi_{i_n}(z(n-1))$ and $\phi_{i_n}(z(n-2)) = \phi_{i_n}(z(n-1))$ by applying EMP twice to two different games with respect to two different solutions. Next, all players $i_1, i_2, \dots, i_{n-1}$ are symmetric in the game $z(n-2)$, whence by ETP and PO it holds that $\psi(z(n-2)) = \phi(z(n-2))$.

By applying a similar reasoning as above, and by employing that i_{n-2} was arbitrarily chosen, we get $\psi(z(n-3)) = \phi(z(n-3))$.

$\vdots$

Then that, $i_2 \in N \setminus \{i_1\}$ was arbitrarily fixed, ψ is defined on B, it is PO and EMP imply that $\psi(z) = \phi(z)$.

Let $k = i_1$, w, $z(i)$ be as in the head of the theorem. Since $\forall i \in N \setminus \{i_1\}$: $w'_i = z(i)'_i$, $\forall z \in B$: $\psi(z) = \phi(z)$, ψ is defined on A, and it is PO and EMP, therefore $\psi(w) = \phi(w)$.

$w'_{i_1} = v'_{i_1}$, and ψ is EMP, hence $\psi_{i_1}(v) = \phi_{i_1}(v)$.

i_1 was arbitrarily fixed, therefore $\psi(v) = \phi(v)$. Q.E.D.

Next, we show that the above theorem implements Young's [8] result.

Theorem 3.4 (Young). *ψ a solution on $\mathcal{G}^N$ is PO, ETP, and EMP if and only if $\psi = \phi$.*

To prove Theorem 3.4. it is enough to show that $\mathcal{G}^N$ is EMP-closed.

Proposition 3.5. *$\mathcal{G}^N$ is EMP-closed.*

First we prove the next lemma.

Lemma 3.6. *Let $v \in \mathcal{G}^N$ be arbitrarily fixed. Then $S \subseteq N$ is an equivalence class in v if and only if $\forall T, Z \subseteq N$ such that $T \setminus S = Z \setminus S$ and $|T| = |Z|$: $v(T) = v(Z)$.*

Proof. if: It is left for the reader.

only if: W.l.o.g. we can assume that $T \setminus Z \neq \emptyset$, and let $T \setminus Z = \{l_1, \dots, l_m\}$ and $Z \setminus T = \{q_1, \dots, q_m\}$. Since $T \setminus Z \subseteq S$ and $Z \setminus T \subseteq S$, S is an equivalence class in v, hence $\forall i$, $1 \leq i \leq m$:

$$\begin{aligned} &v((T \cap Z) \cup \{l_1, \dots, l_i\}) \\ &= v((T \cap Z) \cup \{l_1, \dots, l_{i-1}\}) + v'_{l_i}((T \cap Z) \cup \{l_1, \dots, l_{i-1}\}) \\ &= v((T \cap Z) \cup \{q_1, \dots, q_{i-1}\}) + v'_{q_i}((T \cap Z) \cup \{q_1, \dots, q_{i-1}\}) \\ &= v((T \cap Z) \cup \{q_1, \dots, q_i\}) \end{aligned}$$

Therefore $v(T) = v(Z)$. Q.E.D.

Next, we consider a direct corollary of Lemma 3.6.

Corollary 3.7. *Let $v \in \mathcal{G}^N$ be arbitrarily fixed, $S \subseteq N$ be an equivalence class in v, and $k \in N \setminus S$ be also arbitrarily fixed. Then $\forall T, Z \subseteq N$ such that $T \setminus S = Z \setminus S$ and $|T| = |Z|$: $v'_k(T) = v'_k(Z)$.*

The proof of Proposition 3.5. Let $v \in \mathcal{G}^N$ be such that $S \subset N$ is an equivalence class in v, and $k \in N \setminus S$ be arbitrarily fixed.

If $T = \emptyset$ then let $w(T) = 0$. If $T \cap (S \cup \{k\}) = \emptyset$, $T \neq \emptyset$ then let $w(T)$ be arbitrarily fixed. In the other cases ($T \cap (S \cup \{k\}) \neq \emptyset$), let

$$w(T) \doteq w(T \setminus (S \cup \{k\})) + \sum_{i=1}^{m} v'_k((T \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\}) \ , \qquad (3)$$

where $m = |(S \cup \{k\}) \cap T|$, and $l_i \in S \cap T$, $i = 1, \dots, m-1$. Notice that from Corollary 3.7. $\sum\limits_{i=1}^{m} v'_k((T \setminus (S \cup \{k\})) \cup \{l_1, \dots, l_{i-1}\})$ does not depend on the ordering of the elements of $S \cap T$, i.e. w is well-defined.

It is easy to verify that $w'_k = v'_k$, furthermore, from Lemma 3.6. $S \cup \{k\}$ is an equivalence class in w. Q.E.D.

The proof of Theorem 3.4. See Theorem 3.3. and Proposition 3.5. Q.E.D.

Next, we show that Young's axiomatization is also valid on all the considered subclasses of games.

Theorem 3.8. *ψ a solution on the class of essential / (strictly) convex / (strictly) superadditive / (strictly) weakly superadditive / (strictly) monotonic / additive / (strictly) weakly subadditive / (strictly) subadditive / (strictly) concave games is PO, ETP, and EMP if and only if $\psi = \phi$.*

In order to apply Theorem 3.3 we need the following lemmata (the proofs are relegated to the Appendix):

Lemma 3.9. *The class of the strictly convex / additive / strictly concave games is EMP-closed.*

Lemma 3.10. *Let $v \in \mathcal{G}^N$ be an arbitrarily fixed essential / convex / (strictly) superadditive / (strictly) weakly superadditive / (strictly) monotonic game, and $k \in N$ be arbitrarily fixed. Then $\exists w$ essential / convex / (strictly) superadditive / (strictly) weakly superadditive / (strictly) monotonic game, and $\forall i \in N \setminus \{k\}$: $\exists z(i)$ strictly convex games such that $w'_k = v'_k$ and $\forall i \in N \setminus \{k\}$: $z(i)'_i = w'_i$.*

Lemma 3.11. *Let $v \in \mathcal{G}^N$ be an arbitrarily fixed (strictly) weakly subadditive / (strictly) subadditive/ concave game, and $k \in N$ be arbitrarily fixed. Then $\exists w$ (strictly) weakly subadditive / (strictly) subadditive / concave game, and $\forall i \in N \setminus \{k\}$: $\exists z(i)$ strictly concave games such that $w'_k = v'_k$ and $\forall i \in N \setminus \{k\}$: $z(i)'_i = w'_i$.*

The proof of Theorem 3.8. See Theorem 3.3., and Lemmata 3.9., 3.10., 3.11. Q.E.D.

The importance of Lemmata 3.10. and 3.11. is that not all the considered classes of games are EMP-closed.

Remark 3.12. The proof of Lemma 3.9. shows that the classes of (strictly) convex, (strictly) weakly superadditive, (strictly) monotonic, additive, (strictly) weakly subadditive, (strictly) concave games are EMP-closed. It is common in these games that they can be characterized completely by the properties of the players' marginal contribution functions. This fact is responsible for that their classes are EMP-closed.

However, the classes of essential, (strictly) superadditive, (strictly) subadditive games are not EMP-closed. The next example shows this fact.

Example 3.13. (1) Let $v = (0, 0, 10, 50, 0, 0, 20)$, where $S = \{1, 2\}$ is an equivalence class in v. v is an essential game, however the only game w such that N is an equivalence class in w, and $w'_3 = v'_3$ is $w = (10, 10, 10, 10, 10, 10, -20)$, but that is not essential.

(2) Let $v = (0, 0, 0, 10, 51, 51, 51, 51, 51, 51, 62, 62, 62, 62, 103)$, where $S = \{1, 2, 3\}$ is an equivalence class in v. v is a strictly superadditive game, however the only game w such that N is an equivalence class in w, and $w'_4 = v'_4$, is $w = (10, 10, 10, 10, 61, 61, 61, 61, 61, 61, 72, 72, 72, 72, 113)$, but that is not superadditive. For the subadditive case take $-v$.

Remark 3.14. If $|N| \leq 3$ then the classes of (strictly) superadditive, (strictly) subadditive games coincide with the classes of (strictly) weakly superadditive, (strictly) weakly subadditive games respectively, hence they are EMP-closed. Furthermore, if $|N| = 2$ then the class of essential games coincides with the class of strictly superadditive games, hence it is EMP-closed.

4 Appendix

The proof of Lemma 3.9. Let $v \in \mathcal{G}^N$ be such that $S \subset N$ is an equivalence class in v, and $k \in N \setminus S$ be arbitrarily fixed. The proof of Lemma 3.5. shows that $\exists w \in \mathcal{G}^N$ such that $S \cup \{k\}$ is an equivalence class in w and $w'_k = v'_k$. Moreover, $\forall T$ such that $T \cap (S \cup \{k\}) = \emptyset$, $T \neq \emptyset$: $w(T)$ can be arbitrarily fixed. Therefore, the only thing we have to do (except the trivial case of additive games) is to show that we can give such values to these coalitions that w be in the considered class of games.

(1) The class of additive games: It is well known that game $z \in \mathcal{G}^N$ is additive if and only if $\forall i \in N$: $\exists c_i \in \mathbb{R}$ such that $\forall T \subseteq N \setminus \{i\}$: $z'_i(T) = c_i$.

Let $c^* \doteq v'_k(\emptyset)$. Moreover, $\forall T \subseteq N$ let

$$w(T) = c^* |T| \ . \tag{4}$$

It is easy to see that $w'_k = v'_k$, w is additive, and N is an equivalence class in w.

(2) The class of strictly convex games: Game $z \in \mathcal{G}^N$ is strictly convex if and only if $\forall i \in N$, $\forall T, Z \subseteq N \setminus \{i\}$ such that $Z \subset T$: $z'_i(Z) < z'_i(T)$ (see equation (1)).

Let $M > \max_{T \subset N} |v'_k(T)|$ be arbitrarily fixed. Moreover, let

$$w(T) = M|N|3^{|T|} \ , \tag{5}$$

where $T \cap (S \cup \{k\}) = \emptyset$, $T \neq \emptyset$, and let $w(\emptyset) = 0$.

Let $l \in N \setminus (S \cup \{k\})$ be arbitrarily fixed, and $T, Z \subseteq N \setminus \{l\}$ such that $Z \subset T$. Then

$$\begin{aligned} & w'_l(T) = w(T \cup \{l\}) - w(T) \\ = w((T \cup \{l\}) \setminus (S \cup \{k\})) + & \sum_{i=1}^{m} w'_{l_i}(((T \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\ & - w(T \setminus (S \cup \{k\})) - \sum_{i=1}^{m} w'_{l_i}((T \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \ , \end{aligned}$$

and

$$\begin{aligned} & w'_l(Z) = w(Z \cup \{l\}) - w(Z) \\ = w((Z \cup \{l\}) \setminus (S \cup \{k\})) + & \sum_{i=1}^{n} w'_{l_i}(((Z \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\ & - w(Z \setminus (S \cup \{k\})) - \sum_{i=1}^{n} w'_{l_i}((Z \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \ , \end{aligned}$$

where $m = |(S \cup \{k\}) \cap T|$, $n = |(S \cup \{k\}) \cap Z|$, and $\{l_1, \ldots, l_n\} = (S \cup \{k\}) \cap Z = (S \cup \{k\}) \cap (Z \cup \{l\}) \subseteq (S \cup \{k\}) \cap (T \cup \{l\}) = (S \cup \{k\}) \cap T = \{l_1, \ldots, l_m\}$.

Notice that, if $T \setminus (S \cup \{k\}) = Z \setminus (S \cup \{k\})$, then the proof is complete. Therefore, w.l.o.g. we can assume that $Z \setminus (S \cup \{k\}) \subset T \setminus (S \cup \{k\})$. That v is a strictly convex game, $S \cup \{k\}$ is an equivalence class in w, and $n < |N|$ imply that

$$\begin{aligned} & \sum_{i=1}^{m} w'_{l_i}(((T \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\ & \quad - \sum_{i=1}^{m} w'_{l_i}((T \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\ & - \sum_{i=1}^{n} w'_{l_i}(((Z \cup \{l\}) \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) \\ & + \sum_{i=1}^{n} w'_{l_i}((Z \setminus (S \cup \{k\})) \cup \{l_1, \ldots, l_{i-1}\}) > -2M|N| \ . \end{aligned} \tag{6}$$

On the other hand, from (5)

$$w((T \cup \{l\}) \setminus (S \cup \{k\})) - w(T \setminus (S \cup \{k\})) = 2M|N|3^{|T \setminus (S \cup \{k\})|} ,$$

and

$$w((Z \cup \{l\}) \setminus (S \cup \{k\})) - w(Z \setminus (S \cup \{k\})) = 2M|N|3^{|Z \setminus (S \cup \{k\})|} ,$$

hence $Z \subset T$ implies (remember $Z \setminus (S \cup \{k\}) \subset T \setminus (S \cup \{k\})$)

$$\begin{aligned} & w((T \cup \{l\}) \setminus (S \cup \{k\})) - w(T \setminus (S \cup \{k\})) \\ & -w((Z \cup \{l\}) \setminus (S \cup \{k\})) + w(Z \setminus (S \cup \{k\})) \\ & = 2M|N|3^{|Z \setminus (S \cup \{k\})|}(3^{|(T \setminus Z) \setminus (S \cup \{k\})|} - 1) > 2M|N| . \end{aligned} \tag{7}$$

Summing up (6) and (7)

$$w'_l(T) - w'_l(Z) > 0 . \tag{8}$$

$w'_k = v'_k$, v is strictly convex, $\forall i \in S \cup \{k\}$: $i \sim^w k$, moreover $l \in N \setminus (S \cup \{k\})$ and $T, Z \subseteq N \setminus \{l\}$, $Z \subset T$ were arbitrarily fixed, hence w is strictly convex.

(3) The class of strictly concave games: Game $z \in \mathcal{G}^N$ is strictly concave if and only if $\forall i \in N$, $\forall T, Z \subseteq N \setminus \{i\}$ such that $Z \subset T$: $z'_i(Z) > z'_i(T)$ (see equation (1)).

Equation (2) implies that $\bar{v}$ is a strictly convex game and S is an equivalence class in $\bar{v}$. Then from point (2), $\exists z$ strictly convex game such that $S \cup \{k\}$ is an equivalence class in z, and $z'_k = \bar{v}'_k$. From equation (2) $\bar{z}$ is a strictly concave game, and $S \cup \{k\}$ is an equivalence class in it.

It is a slight calculation to show that $\bar{z}'_k = v'_k$. Finally, let $w = \bar{z}$. Q.E.D.

The proof of Lemma 3.10. Let $M > \max_{T \subset N} |v'_k(T)|$ be arbitrarily fixed. Moreover, let

$$w(T) = 2M|N|3^{|T|} ,$$

where T is such that $k \notin T$, $T \neq \emptyset$, and let $w(\emptyset) = 0$.

If $k \in T$ then let $w(T) = w(T \setminus \{k\}) + v'_k(T \setminus \{k\})$. It is easy to verify that w is an essential / convex / (strictly) superadditive / (strictly) weakly superadditive / (strictly) monotonic game and $w'_k = v'_k$. Furthermore, let $l \in N \setminus \{k\}$, and $T, Z \subseteq N \setminus \{l\}$, $Z \subset T$ be arbitrarily fixed. Then we consider three cases.

$k \in Z$: Then

$$\begin{aligned} & w'_l(T) - w'_l(Z) = w(T \cup \{l\}) - w(T) - w(Z \cup l) + w(Z) \\ & = w((T \setminus \{k\}) \cup \{l\}) + w'_k((T \setminus \{k\}) \cup \{l\}) - w(T \setminus \{k\}) - w'_k(T \setminus \{k\}) \\ & -w((Z \setminus \{k\}) \cup \{l\}) - w'_k((Z \setminus \{k\}) \cup \{l\}) + w(Z \setminus \{k\}) + w'_k(Z \setminus \{k\}) . \end{aligned}$$

Moreover

$$
\begin{aligned}
&w'_k((T \setminus \{k\}) \cup \{l\}) - w'_k(T \setminus \{k\}) \\
&-w'_k((Z \setminus \{k\}) \cup \{l\}) + w'_k(Z \setminus \{k\}) > -4M \ .
\end{aligned} \tag{9}
$$

Furthermore

$$
w((T \setminus \{k\}) \cup \{l\}) - w(T \setminus \{k\}) = 4M|N|3^{|T|-1} \ ,
$$

and

$$
w((Z \setminus \{k\}) \cup \{l\}) - w(Z \setminus \{k\}) = 4M|N|3^{|Z|-1} \ .
$$

Therefore

$$
\begin{aligned}
w((T \setminus \{k\}) \cup \{l\}) - w(T \setminus \{k\}) - w((Z \setminus \{k\}) \cup \{l\}) + w(Z \setminus \{k\}) \\
= 4M|N|3^{|Z|-1}(3^{|T \setminus Z|} - 1) \ .
\end{aligned} \tag{10}
$$

Summing up (9) and (10)

$$
w'_l(T) - w'_l(Z) > 0 \ .
$$

The proofs of the other two cases ($k \notin T$ and $k \in T \setminus Z$) go as above.

Let $i \in N \setminus \{k\}$ be arbitrarily fixed. Then we can repeat the above discussion ($v = w$, $k = i$) to get $z(i)$. Then $z(i)'_i = w'_i$, and $\forall T, Z \subseteq N \setminus \{i\}$ such that $Z \subset T$: $w'_i(T) - w'_i(Z) > 0$ implying that $z(i)$ is strictly convex. Q.E.D.

The proof of Lemma 3.11. The class of the (strictly) weakly subadditive / (strictly) subadditive / concave games contains the class of the strictly concave games. It is a slight calculation to show that we can take the dual of any (strictly) weakly subadditive / (strictly) subadditive / concave game, apply Lemma 3.10., and take the duals of the the games produced by Lemma 3.10. (it is worth noticing that the dual of a (strictly) subadditive / (strictly) weakly subadditive game is not necessarily (strictly) superadditive / (strictly) weakly superadditive, e.g. $v \mathrel{\hat{=}} (4, 4, 4, 4, 4, 4, 7)$ is strictly subadditive, but $\bar{v}$ is not weakly superadditive; moreover, the dual of a (strictly) superadditive / (strictly) weakly superadditive game is not necessarily (strictly) subadditive / (strictly) weakly subadditive, e.g. $v \mathrel{\hat{=}} (0, 0, 0, 3, 1, 2, 4)$ is strictly superadditive, but $\bar{v}$ is not weakly subadditive). Q.E.D.

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