Merging and splitting in cooperative games: some (im)possibility results*

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Abstract

Allocation rules for cooperative games can be manipulated by coalitions merging into single players, or, conversely, players splitting into a number of smaller units. This paper collects some (im)possibility results on merging- and splitting-proofness of (core) allocation rules for cooperative games with side-payments.

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1 Introduction

A cooperative game with side-payments is a very summary description of an underlying game of conflict. It specifies a finite set of players and a worth, in monetary units, for each coalition of players.

In an application of cooperative game theory, the primary problem for an analyst would be to identify the player set, and subsidiarily to determine the worth of each coalition. Players may represent groups of persons, such as labor unions, towns, nations, etc., or they may be other economic variables of the situation under consideration, for example factors of production or objectives of an economic project (Peleg and Sudhölter, 2003, Remark 2.1.3). Since there is likely to be more than one way of fixing the variables of the game, it is fundamental for an analyst to understand if or in which way it matters how the player set itself is determined from the data of the situation.

Players may also be agents who can exit (enter) the game by handing over (receiving) their assets to (from) other agents, or groups of agents can merge and then jointly act as one decision unit, e.g. as a household or a firm. Depending on the specifics of the game and allocation rule, players may have incentives to merge, or to split themselves into smaller units, i.e., the game itself may be subject to strategic manipulation.

Manipulation of allocation rules for cooperative game situations has been a recurrent theme in the literature.[1] In the context of cooperative games with side-payments, the emphasis of the previous research has primarily been on merge properties of probabilistic values.[2] Lehrer (1988) investigates bilateral mergers (called *amalgamations*), where two players merge into one player.

[1] In the context of *bargaining problems*, Harsanyi (1977) discusses the so-called joint-bargaining paradox of the Nash bargaining solution. Harsanyi points out that if two players merge into a single bargaining unit, they tend to weaken their bargaining position. In *bankruptcy problems* conditions similar to the joint properties of merging- and splitting-proofness have been used to characterize the proportional allocation rule, see, for example, Moulin (2002, p. 298) or Thomson (2003, p. 286).

[2] Postlewaite and Rosenthal (1974) is a notable exception.

Lehrer shows that for the Banzhaf value it is always profitable to merge, and he uses this condition for an axiomatic characterization of this value. Haviv (1995) uses a consistency property with respect to consecutive mergers for a characterization of the Shapley value. Derks and Tijs (2000) consider a given partition of the player set and study the game that evolves when the players in each compartment merge into one player. They show that if certain conditions are satisfied then a merger in a given compartment is profitable when players are rewarded according to the Shapley value. Haller (1994) investigates collusion properties of the Shapley value, the Banzhaf value, and other probabilistic values for bilateral *proxy-* and *association agreements*. A bilateral proxy agreement is similar to a bilateral merger, if we disregard null players. A bilateral association agreement modifies the game such that, if just one of the players in the association enters some coalition, then the player's contribution to its worth is as if both players in the association were entering. Segal (2003) obtains complete characterizations of the profitability of three types of integration in a game solved by a probabilistic value.

The present paper considers whether any (core) allocation rule[3] — probabilistic or not — can be merging-proof (i.e., robust against manipulations of the kind where a coalition of players merge into one player) or splitting-proof (i.e., robust against manipulations of the kind where a player is divided into several smaller players), and provides impossibility and possibility results in this direction. The results are collected in Section 2. Briefly, Section 2.1 finds that an anonymous allocation rule cannot simultaneously be merging- and splitting-proof, even if we restrict attention to strictly monotonic convex games (a game is convex if the incentives for joining a coalition increase as the coalition grows, cf. Shapley, 1971). In fact, there exists no splitting-proof anonymous allocation rule on the class of monotonic convex games. On the class of monotonic games with a nonempty core, an allocation rule

[3] An allocation rule specifies, for each game, how the gains from cooperation (that is, the worth of the grand coalition) is distributed among the players. An allocation is in the core if the worth of each coalition does not exceed its aggregate payoff.

can be merging-proof, but we show that then it cannot be a core allocation rule. Likewise, a splitting-proof allocation rule cannot be a core allocation rule. Section 2.2 shows that the Fujishige-Dutta-Ray allocation rule (which selects the most equal allocation in the core) is merging-proof on the class of monotonic convex games. Moreover, we show that there exists a core allocation rule which is splitting-proof on the class of strictly monotonic convex games.

1.1 Definitions and notation

Let $\mathbb{N} = \{1, 2, ...\}$ denote the set of *potential agents*. A *cooperative game with side-payments* is a pair (N, v), where N is a finite set of disjoint nonempty subsets of $\mathbb{N}$ and v is a real-valued function defined on the subsets of N and $v(\emptyset) = 0$. An element of N is called a *player*. Thus, a player consists of one or more potential agents.[4]

An element x of $\mathbb{R}^N$ is a *payoff vector*. For $x \in \mathbb{R}^N$ and $S \subseteq N$ we define $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. If $x(N) = v(N)$ then x is called an *allocation*. The *core* of a game (N, v) is the set $C(N, v) = \{x \in \mathbb{R}^N | x(S) \geq v(S)$ for all $S \subset N$ and $x(N) = v(N)\}$. Note that $\subset$ denotes proper subset.

A game (N, v) is *convex* if $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ for all $T \subset S \subseteq N$, $i \notin S$, it is *superadditive* if $v(S \cup T) \geq v(T) + v(S)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$, it is *monotonic* (*strictly monotonic*) if $v(S \cup \{i\}) \geq (>)$ $v(S)$ for all i and $S \subset N$, $i \notin S$, and *balanced* if $C(N, v) \neq \emptyset$.

An *allocation rule* is a function ϕ that assigns an allocation to any game (N, v). We say that ϕ is a *core allocation rule* if $\phi(N, v) \in C(N, v)$ whenever $C(N, v) \neq \emptyset$. An allocation rule is *anonymous* if it is independent of the names of the players. To be precise, for any game (N, v) and any player set M, if g is a bijective function from N to M and (M, v') is the game defined by $v'(g(S)) = v(S)$ for all $S \subseteq N$, then $\phi_i(N, v) = \phi_{g(i)}(M, v')$ for all $i \in N$.

[4] For a general treatment of cooperative games, see, e.g., Peleg and Sudhölter (2003).

For a game (N, v) and a nonempty coalition $T \subset N$ we define the T-*merger game* (N^T, v^T) as follows: $N^T = \{T\} \cup \{i \in N \mid i \notin T\}$ and $v^T(S) = v(\overline{S})$ for all $S \subseteq N^T$, where $\overline{S} = \{i \in N \mid i \in T \text{ or } i \in S \backslash T\}$ if $T \in S$ and $\overline{S} = S$ otherwise. Note that T is a coalition in (N, v) and a player in the T-merger game (N^T, v^T).[5]

We say that an allocation rule ϕ is *merging-proof (splitting-proof)* on a given family of games if whenever (N, v) and (N^T, v^T) are members of the family and (N^T, v^T) is the T-merger game of (N, v) we have $\phi_T(N^T, v^T) \leq (\geq) \sum_{i \in T} \phi_i(N, v)$. Thus, an allocation rule is merging-proof if the players in a coalition never gain from acting as one player. Splitting-proofness says that regardless of how a player can be split up into a number of smaller players, the player will never gain from doing so. Put differently, an allocation rule is merging-proof if regardless of how a player is able to divide herself into a group of smaller players, doing so is always weakly profitable; and it is splitting-proof if it is always weakly profitable for any given coalition to merge.

2 Results

2.1 Impossibilities

As mentioned in footnote 1, for *bankruptcy problems*[6] conditions similar to the combination of merging- and splitting-proofness have been used to characterize the proportional allocation rule (O'Neill 1982, Chun 1988, de Frutos 1999, Ju 2003) and hence imply anonymity. In that context there ex-

[5] It is readily verified that if $T \subset U \subset N$, then the U-merger game obtained from (N, v) is identical with the $\{T\} \cup \{i \in N | i \notin T, i \in U\}$-merger game obtained from (N^T, v^T). Thus, whether the players in a coalition U merge simultanously or in a sequential manner does not influence the specification of the U-merger game (N^U, v^U).

[6] A *bankruptcy problem* is given by a tuple $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$, $E \leq \sum_{i \in N} c_i$, where c is the vector of claims and E is the estate. A *bankruptcy rule* is a function φ that assigns to every bankruptcy problem a payoff vector $x = \varphi(c, E)$ with $\sum_{i \in N} x_i = E$ and $0 \leq x_i \leq c_i$.

ists large classes of allocation rules that are merging-proof or splitting-proof respectively. We refer to the surveys by Moulin (2002) and Thomson (2003), and the recent contributions by Ju (2003) and Ju et al. (2006), for a detailed account.

In the context of allocation rules for cooperative games, the situation is radically different. Here, the combination of merging- and splitting-proofness does not imply anonymity. In fact, if we restrict attention to the family of strictly monotonic convex games, the combination of merging- and splitting-proofness is inconsistent with anonymity.

Proposition 1 *There is no anonymous merging- and splitting-proof allocation rule on the class of strictly monotonic convex games.*

Proof: By contradiction. Suppose that ϕ is an anonymous, merging- and splitting-proof allocation rule. Let $N = \{1, 2, 3\}$, and let (N, v) be the game defined by $v(\{1\}) = v(\{2\}) = 1$, $v(\{3\}) = 2, v(\{2, 3\}) = v(\{1, 3\}) = 3, v(\{1, 2\}) = 4$ and $v(N) = 6$. We claim that $\phi_1(N, v) = \phi_2(N, v) = \phi_3(N, v) = 2$. For this, notice that by anonymity $\phi_1(N, v) = \phi_2(N, v)$. Moreover, consider a merger by $T = \{1, 2\}$. The resulting game (N^T, v^T) is then defined by $N^T = \{T, 3\}$, $v^T(\{T\}) = 4$, $v^T(\{3\}) = 2$ and $v^T(N^T) = 6$. By merging- and splitting-proofness, we must have $\phi_T(N^T, v^T) = \phi_1(N, v) + \phi_2(N, v)$. Further, for the game (N, w) defined by $w(\{i\}) = 2$ for $i \in N$, $w(\{i, j\}) = 4$ for $i, j \in N, i \neq j$, and $w(N) = 6$, by merging- and splitting-proofness, we have $\phi_1(N, w) + \phi_2(N, w) = \phi_T(N^T, w^T)$. By anonymity, $\phi_i(N, w) = 2$ for $i \in N$ and $\phi_T(N^T, w^T) = 4$, and, because $(N^T, v^T) = (N^T, w^T)$, we have $\phi_1(N, v) + \phi_2(N, v) = 4$. Since $\phi_1(N, v) = \phi_2(N, v)$ and $\phi_1(N, v) + \phi_2(N, v) + \phi_3(N, v) = 6$ we have $\phi_1(N, v) = \phi_2(N, v) = \phi_3(N, v) = 2$, proving the claim.

Now, for the game (N, v), consider the merger of coalition $U = \{2, 3\}$. The resulting game (N^U, v^U) is then given by $N^U = \{1, U\}$, $v^U(\{U\}) = 3$, $v^U(\{1\}) = 1$ and $v^U(N^U) = 6$. We claim that $\phi_U(N^U, v^U) = \frac{18}{4}$ and

$\phi_1(N^U, v^U) = \frac{6}{4}$. For this, let $M = \{1, 2, 3, 4\}$ and consider the game (M, q) with $q(\{i\}) = 1$ for $i \in M$, $q(\{i, j\}) = 2$ for $i, j \in M, i \neq j$, $q(\{i, j, k\}) = 3$ for $i, j, k \in M, i \neq j \neq k$, and $q(M) = 6$. By anonymity we have $\phi_i(M, q) = \frac{6}{4}$ for all $i \in M$. Moreover, let $V = \{2, 3, 4\}$ and consider the V-merger game (M^V, q^V) obtained from (M, q). Note that $M^V = \{1, V\}$, $q^V(\{U\}) = 3$, $q^V(\{1\}) = 1$ and $q^V(M^V) = 6$. Thus, by anonymity and merging- and splitting-proofness we have $\phi_V(M^V, q^V) = \phi_U(N^U, v^U) = \frac{18}{4}$ and $\phi_1(M^V, q^V) = \phi_1(N^U, v^U) = \frac{6}{4}$, proving our claim. We have now obtained a contradiction, since for the game (N, v) the merger $U = \{2, 3\}$ strictly increases aggregate payoff for coalition members.

Finally, we notice that all games that have been considered are convex and strictly monotonic (as can be verified). □

Note that the class of monotonic convex games is a subclass of the monotonic balanced games and the impossibility of Proposition 1 applies therefore to allocation rules defined on this family of games as well.[7] Example 1 shows that anonymity cannot be dispensed with in Proposition 1.

Example 1 Given a player set N, let $i^*(N)$ be the player in N containing the lowest-numbered potential agent. That is, $i^*(N)$ is the player i in N for which: if j is a player in N and a is a potential agent in j, then there is a potential agent b in i such that $b < a$. Note that $i^*(N)$ is well-defined since N consists of disjoint nonempty subsets of $\mathbb{N}$. Then, for an arbitrary class of games, the allocation rule ϕ^* defined by $\phi_i^*(N, v) = v(N)$ if $i = i^*(N)$ and $\phi_i^*(N, v) = 0$ otherwise, is merging- and splitting-proof (as can easily be verified). □

There do exist anonymous merging-proof allocation rules. For example, the *equal split* allocation rule that for any game (N, v) divides $v(N)$ equally among the players is indeed merging-proof on any family of games, but (as

[7] It is easily verified that if (N, v) is a balanced game and if $T \subset N$ is a nonempty coalition, then (N^T, v^T) is balanced.

verified in the first part of Proposition 2 below) it is no coincidence that on balanced games this rule sometimes selects allocations outside the core.

On the class of strictly monotonic games we can find anonymous splitting-proof allocation rules. For example, the allocation rule that for a game (N, v) divides $v(N)$ equally between the players who have the highest single-player worth $v(\{i\})$. Postlewaite and Rosenthal (1974) showed that it is possible to construct a totally balanced[8] (five-player) game (N, v) where the core is a singleton, and in which there is a (three-player) coalition T, such that, if the players in T merge into a single player, then in the core of the T-merger game player T cannot get *more* than what coalition T gets in the core of (N, v) but may get the same or less. Consequently, there exists no core allocation rule on the family of balanced games for which any merger in any game is *strictly* profitable. The first part of Proposition 2 strengthens this observation for anonymous rules.

Proposition 2 *(i) There is no splitting-proof anonymous allocation rule on the class of monotonic convex games. (ii) There is no splitting-proof core allocation rule on the class of monotonic balanced games. (iii) There is no merging-proof core allocation rule on the class of monotonic balanced games.*

Proof: (i). Suppose that ϕ is an anonymous splitting-proof allocation rule. Let $M^T = \{T, 3\}$ denote a player set, where $T = \{1, 2\}$, and define the (monotonic convex) game (M^T, w^T) by $w^T(\{T\}) = w^T(\{3\}) = 0$ and $w^T(M^T) = 1$. By anonymity, $\phi_T(w^T, M^T) = \phi_3(w^T, M^T) = \frac{1}{2}$.

Now, considering the game (M^T, w^T), if splitting T into two players, player 1 and player 2, the (monotonic convex) game (M, w) is obtained with $M = \{1, 2, 3\}$, $w(\{1\}) = w(\{2\}) = w(\{3\}) = w(\{1, 2\}) = w(\{2, 3\}) = w(\{1, 3\}) = 0$ and $w(M) = 1$. Since ϕ is an anonymous allocation rule we must have $\phi_i(M, w) = \frac{1}{3}$ for $i = 1, 2, 3$, contradicting that ϕ is splitting-proof.

[8] A game (N, v) is *totally balanced* if for any nonempty coalition $S \subseteq N$, the game $(S, v_{|S})$ is balanced (where $v_{|S}$ denotes the restriction of v to S).

(ii). Suppose that ϕ is a splitting-proof core allocation rule. Let $M = \{1, ..., 6\}, T = \{1, 2, 3\}$ and $U = \{4, 5, 6\}$. For the set $M^{T^U} = \{T, U\} = M^{U^T}$, define the (monotonic superadditive balanced) game (M^{T^U}, w^{T^U}) as follows: $w^{T^U}(\{T\}) = w^{T^U}(\{U\}) = 2$ and $w^{T^U}(M^{T^U}) = 5$. For either $V = T$ or $V = U$, we have $\phi_V(w^{T^U}, M^{T^U}) \leq \frac{5}{2}$. We assume that $\phi_T(w^{T^U}, M^{T^U}) \leq \frac{5}{2}$. The case $\phi_U(w^{T^U}, M^{T^U}) \leq \frac{5}{2}$ is similar, and thus can be omitted.

For the player set $M^U = \{1, 2, 3, U\}$ define the (monotonic superadditive balanced) game (M^U, w^U) by $w^U(\{1\}) = w^U(\{2\}) = w^U(\{3\}) = 0$, $w^U(\{U\}) = 2$, $w^U(\{1, 2\}) = w^U(\{2, 3\}) = w^U(\{1, 3\}) = 2$, $w^U(\{1, U\}) = w^U(\{2, U\}) = w^U(\{3, U\}) = 3$, $w^U(\{1, 2, 3\}) = 2$, $w^U(\{i, j, k\}) = 3$ for any other three-player coalition in M^U, and $w^U(M^U) = 5$. Note that (M^{T^U}, w^{T^U}) is the T-merger game of (M^U, w^U). Then, $C(M^U, w^U)$ is a singleton; it is the element in $\mathbb{R}^{M^U}$ given by $x_U = 2$ and $x_1 = x_2 = x_3 = 1$. Since ϕ is a core allocation rule we have $\phi_i(M^U, w^U) = 1$ for $i = 1, 2, 3$, and $\phi_U(M^U, w^U) = 2$ contradicting that ϕ is splitting-proof.

(iii). Suppose that ϕ is a merging-proof core allocation rule. Let $N = \{1, 2, 3, 4\}$, $T = \{1, 2\}$ and $U = \{3, 4\}$, such that we have $N^T = \{T, 3, 4\}$ and $N^U = \{1, 2, U\}$. We define the game (N^T, v^T) as follows: $v^T(\{i\}) = 0$ for all $i \in N^T$, $v^T(\{3, 4\}) = 0$, $v^T(\{T, 3\}) = v^T(\{T, 4\}) = 1$ and $v^T(N^T) = 1$. Then, $C(N^T, v^T)$ is a singleton; it is the element in $\mathbb{R}^{N^T}$ given by $x_T = 1$ and $x_3 = x_4 = 0$. Since ϕ is a core allocation rule we have $\phi_T(N^T, v^T) = 1$ and $\phi_3(N^T, v^T) = \phi_4(N^T, v^T) = 0$.

Let (N^{T^U}, v^{T^U}) denote the U-merger game of (N^T, v^T). Thus, $N^{T^U} = \{T, U\}$, $v^{T^U}(\{T\}) = 0, v^{T^U}(\{U\}) = 0$ and $v^{T^U}(N^{T^U}) = 1$. Since ϕ is merging-proof and a core allocation rule, we have $\phi_T(N^{T^U}, v^{T^U}) = 1$ and $\phi_U(N^{T^U}, v^{T^U}) = 0$.

For the set N^U define the game (N^U, v^U) as follows: $v^U(\{i\}) = 0$ for all $i \in N^U$, $v^U(\{1, 2\}) = 0$, $v^U(\{1, U\}) = v^T(\{2, U\}) = 1$ and $v^U(N^U) = 1$. Then, $C(N^U, v^U)$ is a singleton; it is the element in $\mathbb{R}^{N^U}$ given by $x_U = 1$ and $x_1 = x_2 = 0$. Since ϕ is a core allocation rule we have $\phi_U(N^U, v^U) = 1$

and $\phi_1(N^U, v^U) = \phi_2(N^U, v^U) = 0$.

Now, let (N^{U^T}, v^{U^T}) denote the T-merger game of (N^U, v^U). Since ϕ is a merging-proof core allocation rule, we have $\phi_U(N^{U^T}, v^{U^T}) = 1$ and $\phi_T(N^{U^T}, v^{U^T}) = 0$ — a contradiction since the games (N^{U^T}, v^{U^T}) and (N^{T^U}, v^{T^U}) are identical. □

We notice that the proof of part (iii) of Proposition 2, and by implication from convexity also part (i), involves only (monotonic superadditive) totally balanced games. The proof of part (ii) relies on games there are superadditive but not totally balanced.

2.2 Possibilities

For the family of probabilistic values, Haller (1994, Corollary 3.3) gives sufficient conditions for which bilateral proxy agreements are always weakly (un)profitable. The Shapley value does not satisfy these conditions,[9] and core compatibility was not addressed in Haller's study. Indeed, the Shapley value is neither merging-proof, nor splitting-proof, even on the class of strictly monotonic convex games, as verified in Example 2 below. Note that bilateral merging-proofness (i.e., the property that a T-merger is not strictly profitable if $|T| = 2$) does not imply merging-proofness. An analogous statement holds for splitting-proofness.

Example 2 Let (N, v) be the (strictly monotonic convex) game, where $N = \{1, 2, 3, 4\}$ and v is given by $v(S) = 1$ if $|S| = 1$, $v(S) = 3$ if $|S| = 2$, $v(S) = 6$ if $|S| = 3$ and $v(N) = 9$. The Shapley value is $\phi_i^{Sh}(N, v) = \frac{9}{4}$ for all $i \in N$. Now for $T = \{3, 4\}$ consider the T-merger game where $N^T = \{1, 2, T\}$

[9] The Shapley value ϕ^{Sh} can be defined as

$$\phi_i^{Sh}(N, v) = \sum_{S \subseteq N, S \ni i} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!}(v(S) - v(S \backslash \{i\})).$$

and v^T takes the following values: $v^T(\{1\}) = v^T(\{2\}) = 1$, $v^T(\{T\}) = 3$, $v^T(\{1,2\}) = 3$, $v^T(\{1,T\}) = v^T(\{2,T\}) = 6$ and $v^T(N^T) = 9$. Then, $\phi_T^{Sh}(N^T, v^T) = \frac{14}{3} > \frac{9}{2}$. Thus, for players 3 and 4 (or any other two player coalition), merging is strictly profitable.

Next, consider the (strictly monotonic convex) game (M, w) where $M = \{1,2,3\}$, $w(S) = |S|$ if $|S| < 3$ and $w(M) = 4$. Then, $\phi_i^{Sh}(M, w) = \frac{4}{3}$ for all i. For $T = \{1,2\}$, the T-merger game w^T is defined by $M^T = \{T, 3\}$, $w^T(\{T\}) = 2$, $w^T(\{3\}) = 1$ and $w^T(M^T) = 4$. Then, $\phi_T^{Sh}(M^T, w^T) = \frac{5}{2} < \frac{8}{3}$. Thus, splitting T is strictly profitable for the potential agents 1 and 2. □

The *nucleolus* (Schmeidler 1969) is the allocation rule ϕ^{Nu} that to each game (N, v) assigns an allocation $x = \phi^N(N, v)$ such that x lexicographically minimizes the vector of excesses $e(S, x) = v(S) - \sum_{i \in S} x_i$, $\emptyset \subset S \subset N$, when these are arranged in order of descending magnitude. The *per capita nucleolus* ϕ^{PCNu} (Grotte, 1970) is the analog of the nucleolus with excesses defined on a per capita basis: $e(S, x) = \frac{v(S) - \sum_{i \in S} x_i}{S}$, $\emptyset \subset S \subset N$ (see, e.g., Young 1985). We can use an example discussed in Hokari (2000) to show that none of these allocation methods are merging-proof:

Example 3 Let $N = \{1,2,3,4\}$, and let (N, v) be monotonic convex game defined by $v(\{i\}) = 0$ for all i, $v(\{1,3\}) = 0$, $v(\{1,2\}) = v(\{1,4\}) = v(\{2,3\}) = v(\{2,4\}) = v(\{3,4\}) = 2$, $v(\{1,2,3\}) = 4$, $v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 6$, and $v(\{N\}) = 10$. Hokari (2000) shows that $\phi^{Nu}(N, v) = (2,2,2,4)$. It is easy to verify that $\phi^{PCNu}(N, v) = \phi^{Nu}(N, v)$.

Now, consider the merger of coalition $T = \{1,2,3\}$. The T-merger game (N^T, v^T) is given by $N^T = \{T, 4\}$, $v^T(\{T\}) = 4$, $v^T(\{4\}) = 0$ and $v^T(N^T) = 10$. We then have $\phi^{Nu}(N^T, v^T) = \phi^{PCNu}(N^T, v^T) = (7,3)$. Thus, the merger is strictly profitable.[10] □

[10] From Proposition 1(i) we further know that we can find a monotonic convex game, such that a split is strictly profitable. Since the nucleolus is continuous in v (cf. Peleg and Sudhölter, 2003, Chapter 9), we can infer that it fails merging-proofness and splitting-proofness on the family of *strictly* monotonic convex games as well. The same conclusion

Examples 2 and 3 illustrate that merging-proofness is, indeed, a very restrictive requirement. Thus, one may wonder whether there exists *any* merging-proof core allocation rule even on convex games. Lemma 1 below verifies that the family of convex games is closed under mergers, so requiring merging-proofness on this domain imposes a profitability restriction on *all* possible T-mergers in any convex game.

Lemma 1 *Let (N, v) be a (monotonic) (strictly monotonic) convex game and $T \subset N$ a nonempty coalition. Then, the T-merger game $\left(N^T, v^T\right)$ is a (monotonic) (strictly monotonic) convex game.*

Proof: Suppose that (N, v) is convex. Let $S, S' \subseteq N^T$. First, we claim that $\overline{S \cap S'} = \overline{S} \cap \overline{S'}$. For this, consider a player $i \in N \backslash T$. Then, $i \in \overline{S \cap S'}$ if and only if [$i \in S$ and $i \in S'$] if and only if [$i \in \overline{S}$ and $i \in \overline{S'}$]. Further, consider the player T in N^T. Then, for any $i \in T \subset N$, $i \in \overline{S \cap S'}$ if and only if [$T \in S$ and $T \in S'$] if and only if [$i \in \overline{S}$ and $i \in \overline{S'}$], which proves the claim.

Second, we claim that $\overline{S \cup S'} = \overline{S} \cup \overline{S'}$. The claim is verified in a similar way: Consider a player $i \in N \backslash T$. Then, $i \in \overline{S \cup S'}$ if and only if [$i \in S$ or $i \in S'$] if and only if [$i \in \overline{S}$ or $i \in \overline{S'}$]. Further, consider the player T in N^T. Then, for any $i \in T \subset N$, $i \in \overline{S \cup S'}$ if and only if [$T \in S$ or $T \in S'$] if and only if [$i \in \overline{S}$ or $i \in \overline{S'}$], which proves the claim.

The game (N^T, v^T) is convex if

$$v^T(S \cap S') + v^T(S \cup S') \geq v^T(S) + v^T(S') \text{ for all } S, S' \subseteq N^T,$$

(see, e.g., Peleg and Sudhölter, 2003, p. 13), i.e., if

$$v(\overline{S \cap S'}) + v(\overline{S \cup S'}) \geq v(\overline{S}) + v(\overline{S'}) \text{ for all } S, S' \subseteq N^T.$$

But, since $v(\overline{S \cap S'}) = v(\overline{S} \cap \overline{S'})$ and $v(\overline{S \cup S'}) = v(\overline{S} \cup \overline{S'})$, this is equivalent

holds for the per capita nucleolus.

to

$$v(\overline{S}\cap\overline{S'})+v(\overline{S}\cup\overline{S'})\geq v(\overline{S})+v(\overline{S'}) \text{ for all } S,S'\subseteq N^T,$$

which is satisfied since (N,v) is convex and $\overline{S},\overline{S'}\subseteq N$.

Finally, we notice that since a convex game (N,v) is superadditive, it is monotonic (strictly monotonic) if and only if $v(\{i\})\geq 0$ $(v(\{i\})>0)$ for all $i\in N$. This condition implies that $v^T(\{i\})\geq 0$ $(v^T(\{i\})>0)$ for all $i\in N^T$ since $v(T)\geq v(\{i\})$ for all $i\in T$, i.e., (N^T,v^T) is also monotonic (strictly monotonic). □

Dutta and Ray (1989) showed that in the core of a convex game there is one and only one allocation which is more equal than any other core allocation, in a Lorenz sense. The allocation rule that for any convex game selects this allocation was introduced independently by Fujishige (1980) and Dutta and Ray (1989). Suppose that f is a strictly concave function on $\mathbb{R}$. By a result in Hardy et al. (1934, Theorem 108) (see, e.g., Hougaard, Peleg and Thorlund-Petersen, 2001), we can define the *Fujishige-Dutta-Ray allocation rule* ϕ^{FDR} on the class of convex games as follows: $\phi^{FDR}(N,v)$ is the maximizer of $\sum_{i\in N}f(x_i)$ subject to the constraint $x\in C(N,v)$.

For bankruptcy problems, the constrained equal awards bankruptcy rule is merging-proof (de Frutos 1999, Ju 2003). As pointed out in Thomson (2003), the allocation chosen by the constrained equal awards rule corresponds to the payoff vector chosen by the *Fujishige-Dutta-Ray allocation rule* for the associated *bankruptcy game*.[11] Thus, the *Fujishige-Dutta-Ray allocation rule* is merging-proof on the class of bankruptcy games. We shall prove the following more general result:

Proposition 3 *On the class of monotonic convex games, the Fujishige-Dutta-Ray allocation rule is merging-proof.*

[11] Given a bankruptcy problem $(c,E)\in\mathbb{R}^N_+\times\mathbb{R}_+$, the *bankruptcy game* is the game $(N,v_{(c,E)})$ defined as $v_{(c,E)}(S)=\max\left\{0,E-\sum_{i\in N\setminus S}c_i\right\}$ for all $S\subseteq N$. The class of such bankruptcy games is a subclass of the monotonic convex games (Curiel et al. 1987).

For the proof of Proposition 3, we formulate a lemma which says that it is possible to go from one core allocation to another by a sequence of bilateral transfers (which can be ordered in certain ways) for which any intermediate allocation is also in the core. We make use of the following concepts and definitions (see also Hougaard and Østerdal, 2008). Suppose that $x, y \in \mathbb{R}^N$, and for some $i, j \in N$ and some $\gamma_{ij} \geq 0$, we have $y_i + \gamma_{ij} = x_i$, $y_j - \gamma_{ij} = x_j$ and $x_k = y_k$ for $k \neq i, j$. We then say that y is reached from x after a *bilateral transfer* (of the amount γ_{ij}) from player i to j. A *transfer matrix* is a nonnegative $|N| \times |N|$ matrix $\Gamma = \left[\gamma_{ij}\right]_{i,j \in N}$ such that, if $\gamma_{ij} > 0$ then $\gamma_{j'i} = 0$ for all $j' \in N$. Thus, a transfer matrix induces a tri-partition of N in *payers*, *receivers*, and *unaffected* players. For a finite set D (which could, for example, be a player set or a set of bilateral transfers), a bijective function $\sigma : D \rightarrow \{1, ..., |D|\}$ is an *ordering* (of D). If $d, e \in D$ and $\sigma(d) < \sigma(e)$, we say that d is before e (according to σ).

Lemma 2 *Let (N, v) be a convex game and $x, y \in C(N, v)$. Then, there is a transfer matrix $\Gamma = \left[\gamma_{ij}\right]_{i,j \in N}$ leading from x to y and an ordering σ of the bilateral transfers γ_{ij} in Γ such that after each bilateral transfer the resulting allocation is in $C(N, v)$. In particular, for any given ordering τ of the receivers (payers) i, the ordering σ can be chosen such that, if $\tau(i) < \tau(j)$ then all transfers to (from) player i are made before any bilateral transfer to (from) player j is made.*

Proof: Let $P = \{i | x_i > y_i\}$ and $R = \{i | x_i < y_i\}$ denote the sets of payers and receivers respectively.

First, we claim that for any player i in P, we can always find some player j in R such that the transfer of some amount $0 < \varepsilon_{ij} \leq \min\{x_i - y_i, y_j - x_j\}$ from i to j leads to a new allocation which is also in $C(N,v)$. For this, consider a player $i \in P$, and suppose to the contrary that there is no player j in R for which there can be transferred some amount $0 < \varepsilon_{ij} \leq \min\{x_i - y_i, y_j - x_j\}$ from i to j (upholding the core constraints). This means that for any $j \in R$, there must be a zero-excess coalition S^j at x (i.e., $x(S^j) = v(S^j)$) for which $i \in S^j$ and $j \notin S^j$. By Shapley (1971), the set of zero-excess coalitions in a convex game is a ring (i.e., closed under union and intersection). In particular, $\cap_{j\in R} S^j$ is a zero-excess coalition and note that $i \in \cap_{j\in R} S^j$. Since $i \in \cap_{j\in R} S^j$ and since the set $\cap_{j\in R} S^j$ has empty intersection with R, it contradicts that y is a core allocation (because if $\cap_{j\in R} S^j$ is a zero-excess coalition at x we would have $y(\cap_{j\in R} S) < v(\cap_{j\in R} S)$).

Second, we claim that for an arbitrary player j in R, we can always find some player i in P such that the transfer of some amount $0 < \varepsilon_{ij} \leq \min\{x_i - y_i, y_j - x_j\}$ is possible (upholding the core constraints). For this, consider a player $j \in R$, and suppose to the contrary that there is no player i in P for which there can be transferred some amount $0 < \varepsilon_{ij} \leq \min\{x_i - y_i, y_j - x_j\}$ from i to j. This means that for any $i \in P$, there must be a zero-excess coalition S^i at x for which $i \in S^i$ and $j \notin S^i$. Since $\cup_{i\in P} S^i$ is then also a zero-excess coalition, $P \subseteq \cup_{i\in P} S^i$ and $j \notin \cup_{i\in P} S^i$ it contradicts that y is a core allocation (because if $\cup_{i\in P} S^i$ is a zero-excess coalition at x we would have $y(\cup_{i\in P} S^i) < v(\cup_{i\in P} S^i)$).

To complete the proof, we will show that we can obtain y from x by a finite number of any such bilateral transfers. For this, it is sufficient to show that for any $x, y \in C(N, v)$ and sets P and R as described, any player $i \in P$ can transfer a total amount $x_i - y_i$ to players in R in at most $|R|$ steps (upholding the core constraints in each step). The argument showing that any player $j \in R$ can obtain a total amount of $y_j - x_j$ from players in P in at most $|P|$ steps (upholding the core constraint in each step) is similar and thus can be omitted.

Consider therefore an arbitrary player $i \in P$, and let m_i, $0 < m_i \leq x_i - y_i$, denote the supremum of the total amounts of payoff that can be transferred from player i to one or more players in R by an ordered (finite or countable infinite) sequence of core compatible bilateral transfers, such that each receiver j does not receive more than $y_j - x_j$ and i does not pay more than $x_i - y_i$. Denote the final allocation obtained in the limit of such a sequence of bilateral transfers with y'. First, we will observe that the same final allocation y' can be obtained by an ordered sequence of at most $|R|$ transfers. Second, we will observe that we cannot have $m_i < x_i - y_i$. Note that the combination of these two observations will complete the proof.

To verify the first observation, let m_{ij}, $0 \leq m_{ij} \leq m_i$, denote the supremum of the total amount transferred from i to j. Since $C(N, v)$ is a closed set, the allocation y' is in the core. In particular, we can transfer the entire amounts m_{ij} from i to j in an arbitrary sequence of bilateral transfers involving at most $|R|$ steps. The reason is that if the core constraint for a coalition S, $i \in S$, were violated after some step, then the final allocation would also violate this constraint for coalition S — a contradiction.

To verify the second observation, note that $y' \in C(N, v)$ implies, by a previous argument, that there is an additional core-compatible bilateral transfer from i to some j player in R for which $y'_j < y_j$ — a contradiction. □

We are now ready to prove Proposition 3.

Proof of Proposition 3: Let (N, v) be a convex game, and let $x = \phi^{FDR}(N, v)$. Let $T \subset N$, $T \neq \emptyset$, and let $y = \phi^{FDR}(N^T, v^T)$. Note that $x \in C(N, v)$ and $y \in C(N^T, v^T)$, and thus $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^{N^T}$. We want to show that $x(T) \geq y_T$. For this, we shall demonstrate that if $x(T) < y_T$, then x cannot be chosen by the Fujishige-Dutta-Ray allocation rule for the game (N, v) — a contradiction.

From x define the following allocation $\widetilde{x}$ in $\mathbb{R}^{N^T}$: $\widetilde{x}_T = x(T)$ and $\widetilde{x}_i = x_i$ for $i \in N^T \backslash \{T\}$. We have $\widetilde{x}(N) = v(N)$, and for any coalition $S \subseteq N^T$ we have $\widetilde{x}(S) = x(\overline{S}) \geq v(\overline{S}) = v^T(S)$. Thus, $\widetilde{x} \in C(N^T, v^T)$.

We define the following two sets of players in N^T: $P = \{i \in N^T \backslash \{T\} | y_i < x_i\}$ and $R = T \cup \{i \in N^T \backslash \{T\} | y_i > x_i\}$. Hence, in $C(N^T, v^T)$ we can obtain y from $\widetilde{x}$ by bilateral transfers from players in P to players in R. By Lemma 2, there exists a sequence of these bilateral transfers, such that after each step in this sequence, the allocation obtained is in $C(N^T, v^T)$ and player T first begins to receive payoff from the players P when all other players in R have obtained all their payoff (i.e., each player $i \in R \backslash T$ has received $y_i - x_i$). Given this sequence of bilateral transfer, let P' denote the subset of players in P which transfer a positive amount of payoff to player T. By Lemma 2, these bilateral transfers from players in P' to player T can be made in an arbitrary order (upholding the core constraints before and after each step). Hence, each of these transfers from players in P to player T must increase the value of the allocation measured by the objective function $\sum_{i \in N^T} f$ on $\mathbb{R}^{N^T}$.

Consider now the game (N, v) and $C(N, v)$. Since (N, v) and (N^T, v^T) are monotonic games, $C(N, v) \subseteq \mathbb{R}_+^N$ and $C(N^T, v^T) \subseteq \mathbb{R}_+^{N^T}$. Thus, $\widetilde{x}_T \geq x_i$ for all $i \in T$. In particular, since f is strictly concave, for any player i in P' it follows that there is a (sufficiently small) amount of payoff p_i such that a bilateral transfer of p_i from i to any player in T increases the value of the allocation measured by the objective function $\sum_{i \in N} f$ on $\mathbb{R}^N$. Since $x = \phi^{FDR}(N, v)$ any such transfer must violate a core constraint. In particular, for an arbitrary player $i \in P'$, for any player j in T there must be a zero-excess coalition $S^j \subseteq N$ at x such that $i \in S^j$ and $j \notin S^j$. Hence, $i \in \cap_{j \in T} S^j \subseteq N \backslash T$ and $\cap_{j \in T} S^j$ is a zero-excess coalition (since the set of zero-excess coalitions is a ring), contradicting that y is in $C(N^T, v^T)$ since $v^T(\cap_{j \in T} S^j) = v(\cap_{j \in T} S^j) = x(\cap_{j \in T} S^j) > y(\cap_{j \in T} S^j)$. □

Example 4 below shows that monotonicity cannot be dispensed with in Proposition 3. It shows, more generally, that there does not exist an anonymous allocation rule on the class of (not necessarily monotonic) convex games.

Example 4 Suppose that ϕ is an anonymous allocation rule. Let (N, v) be the convex game, where $N = \{1, 2, 3\}$ and v is given by $v(S) = -2$ if $S \neq \emptyset, N$, and $v(N) = -1$. By anonymity, $\phi_i(N, v) = -\frac{1}{3}$ for all $i \in N$.

Let $T = \{1, 2\}$, and consider the T-merger game (N^T, v^T) (with $N^T = \{T, 3\}$, $v^T(\{T\}) = -2$, $v^T(\{3\}) = -2$ and $v^T(N^T) = -1$). By anonymity, $\phi_i\left(N^T, v^T\right) = -\frac{1}{2}$ for all $i \in N^T$. Thus, the T-merger is strictly profitable. $\square$

Proposition 2(i) observed that there is no splitting-proof anonymous allocation rule on the class of monotonic convex games. For the family of *strictly* monotonic convex games, such allocation rules do, in fact, exist. The proof of Proposition 4 is constructive. It specifies an anonymous splitting-proof *core* allocation rule.

Proposition 4 *On the class of strictly monotonic convex games, there exists an anonymous splitting-proof core allocation rule.*

Proof: We define a core allocation rule, called ϕ^*, and show that a merger is always weakly profitable; that is, for any strictly monotonic convex games (N, v) and any $T \subset N$, $T \neq \emptyset$, then $\phi^*_T(N^T, v^T) \geq \sum_{i \in T} \phi^*_i(N, v)$.

For any game (N, v), there is $1 \leq k \leq |N|$ and a partition $P_1, ..., P_k$ of N, classifying players according to non-decreasing contribution to the grand coalition, i.e., for any $1 \leq m < n \leq k$, if $i \in P_m$ and $j \in P_n$ then $v(N) - v(N \backslash \{i\}) < v(N) - v(N \backslash \{j\})$.

Let σ be an ordering of the players in N such that i is before j if there is $m < n$ where $i \in P_m$ and $j \in P_n$. Given σ, let

$$p(\sigma) = (v(\{i\}), v(\{i, j\}) - v(\{i\}), v(\{i, j, h\}) - v(\{i, j\}), ...)$$

be the vector (in $\mathbb{R}^N$) of marginal contributions associated with the ordering σ. We define ϕ^* to be the center of gravity of the $|P_1|!|P_2|! \cdots |P_k|!$ vectors of marginal contributions that can be generated by all such orderings σ, i.e.,

$$\phi^*(N, v) = \frac{p(\sigma^1) + p(\sigma^2) + ...}{|P_1|!|P_2|! \cdots |P_k|!},$$

where $\sigma^1, \sigma^2, ...$ are all possible orderings ranking players according to non-decreasing contribution to the grand coalition as described above (where each ordering appears precisely one time).

Consider the allocation rule ϕ^*. We claim that for any $T \subset N$, $T \neq \emptyset$, a T-merger is always weakly profitable for the players in T.

Note that by strict monotonicity, $v(N) - v(N \backslash \{i\}) > 0$ for each player i in coalition T in the game (N, v). For $1 \leq m \leq k$, let $T \cap P_m = \{i \in N | i \in T$ and $i \in P_m\}$. Let $\lambda_{T \cap P_m} \geq 0$ be the aggregate payoff in the game (N, v) to the players in $T \cap P_m$ when these players occupy the last possible positions taken over all orderings σ as described above (i.e., $\lambda_{T \cap P_m}$ is the sum of the marginal contributions of the players in $T \cap P_m$ when these take the last $|T \cap P_m|$ positions among the $|P_m|$ positions available in an ordering σ ranking players according to their contribution to the grand coalition). Note that the sum of marginal contributions of the players in $T \cap P_m$ does not depend on which specific positions they occupy, as long as they together occupy the last $|T \cap P_m|$ positions among the $|P_m|$ possible positions, so $\lambda_{T \cap P_m}$ is well-defined. We then have

$$\sum_{i \in T} \phi_i^*(N, v) \leq \sum_{m=1,...,k} \lambda_{T \cap P_m}.$$

Now, consider an arbitrary player i in $N \backslash T$ which belongs to a set P_h for which $P_h \cap T \neq \emptyset$ or for which there is some $j \in T$ and $g > h$ such that $j \in P_g$. (Note that P_h and T are subsets of N). Then, in the partition $\widehat{P}_1, ..., \widehat{P}_{k'}$ of N^T, classifying players in the T-merger game (N^T, v^T) according to their contribution to the grand coalition, there is $g' > h'$ such that $i \in \widehat{P}_{h'}$ and $T \in \widehat{P}_{g'}$, since $v^T(N^T) - v^T(N^T \backslash \{T\}) > v(N) - v(N \backslash \{j\})$ for all $j \in T$. (Note that no player in T is a null-player in (N, v), and $v^T(N^T) - v^T(N^T \backslash \{i\}) = v(N) - v(N \backslash \{j\})$ for all $j \in N \backslash T$). Since this holds for any player i in $N \backslash T$, for the T-merger game (N^T, v^T) we have

$$\phi_T^*(N^T, v^T) \geq \sum_{m=1,...,k} \lambda_{T \cap P_m},$$

because by convexity every marginal contribution for player T in (N^T, v^T), for which the players are ordered according to non-decreasing contribution to the grand coalition, is greater than or equal to $\sum_{m=1,\dots,k} \lambda_{T\cap P_m}$. This proves our claim. $\square$

2.3 Final remarks and open questions

Two specific existence problems remains unanswered. We do not know whether there exists a splitting-proof core allocation rule on the class of (not necessarily strictly) monotonic convex games, and we do not know whether there exists a merging-proof core allocation rule on the class of (not necessarily monotonic) convex games. But, in both cases we know that any such allocation rule would fail to be anonymous.

The merging-proofness property of the Fujishige-Dutta-Ray allocation rule appeared to be closely connected to the defining property of this allocation rule of selecting the most equal allocation subject to the core constraints. We leave it as an open question whether the Fujishige-Dutta-Ray allocation rule is the *only* anonymous merging-proof core allocation rule on the class of monotonic convex games.

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