Core-stable rings
in second price auctions with common values

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Abstract

In a common value auction in which the information partitions of the bidders are connected, all rings are core-stable.

1 Introduction

Collusion in auctions is the topic of a number of empirical and theoretical papers (see, e.g., Klemperer (2004), Krishna (2002), Milgrom (2004) for references). While auctions with common values prevail in the empirical studies (see, e.g., Pesendorfer (2000), Porter and Zona (1993)) most theoretical articles focus on the independent private values case (see, e.g., Caillaud and Jehiel (1998), Graham and Marshall (1987), Graham et al. (1990), Lopomo et al. (2005), Mc Afee and Mc Millan (1992), Mailath and Zemsky (1991), Marshall and Marx (2007), Marshall et al. (1994), Waehrer (1999)[1]). Furthermore, collusion in auctions is typically viewed as a mechanism design problem for a given ring, which often involves all the bidders. This approach emphasizes the role of incentive compatibility and individual interim participation constraints, but does not question the participation of subgroups of bidders.

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[1]Lyk-Jensen (1997), Marshall and Meurer (2001) and Mc Afee and Mc Millan (1992) give insights on collusion in auctions with common values.

In this paper, we analyze the stability of rings in a general model of auctions with common values, which has been introduced by Einy et al. (2002). As standard in information economics, the value of the fundamentals (namely, the common value of the object for sale here) depends on a state of nature and the private information of every agent is modeled by a partition of the set of states of nature. Einy et al. further assume that these partitions are connected with respect to the common value (i.e., if a bidder considers two values as possible given his information, he also considers the intermediate values as possible). In this setup, Einy et al. show that second price auctions are dominance solvable. Among the corresponding "sophisticated equilibria", they identify a solution with remarkable properties.

In order to capture participation constraints of subrings, we use a "bridge approach" according to A. Kalai and E. Kalai (2009)'s terminology, i.e., we associate a cooperative game with the noncooperative second price auction. We first show that, in spite of the strategic externalities which are part of the noncooperative game, the cooperative one can be described by a well-founded characteristic function, which does not rely on incredible threats from coalitions. More precisely, rings form ex ante, i.e., before the players learn their private information on the value of the object. This timing is consistent with rings observed in practice (see, e.g., Graham and Marshall (1987)). Members of the ring fully commit to bid as recommended, which amounts to allowing a benevolent "ring center" to make the bids. Leaving aside the problem of information revelation inside the ring, a natural bidding strategy for the ring center is the strategy which is part of the solution identified Einy et al. It turns out that this strategy does not depend on the underlying coalition structure. We further show that, by adopting an appropriate allocation rule, the ring can make this strategy incentive compatible even if information on the state of nature is unverifiable or "soft". In particular, the grand coalition can achieve the first best Pareto optimum.

To sum up, we construct a characteristic function, in which the worth of a ring is its expected payoff in Einy et al.'s solution. A ring is stable if all its subrings agree to participate, namely if the ring can propose a core allocation to its members or, equivalently, if the core of the characteristic function, restricted to the ring, is not empty. Our main result is that, in the second price auctions modeled by Einy et al., all rings are stable in this sense. The same property holds in the case of independent private values (see Mailath and Zemsky (1991)) but, as we shall illustrate, need not hold in common values models in which the bidders' information partitions are not connected. In this case, the cooperative game between the rings may be better described by a partition form in the sense of Lucas and Thrall (1963) (see Barbar and Forges (2007)).

Let us come to the organization of the paper. In section 2, we recall the main features of Einy et al.'s model. In section 3, we introduce an auxiliary noncooperative second price auction, in which the players are bidding rings; we derive some properties of Einy et al.'s solution in the auxiliary game; we show in particular that this solution is incentive compatible. In section 4, we construct the cooperative game in which core-stability can be defined and establish that all rings are core-stable. Section 5 contains some examples and draws some practical conclusions from our analysis.

2 Auctions without collusion

2.1 Basic game

As in Einy et al. (2002), let $N = \{1, ..., n\}$, $n \geq 2$, be the set of bidders, Ω be the finite set of states of nature and p be a probability distribution on Ω (w.l.o.g., $p(\omega) > 0$ for every $\omega \in \Omega$). The bidders participate in a second price auction to acquire a single object. The value of the object $v(\omega)$ is the same for all bidders and depends on the state of nature ω. The private information of bidder i, $i = 1, ..., n$, is described by a partition Π_i of Ω.

The auction game, which we denote as G, starts with a move of nature choosing ω in Ω according to p. Every bidder i ($i = 1, ..., n$) is informed of the element $\pi_i(\omega)$ of Π_i which contains ω and then makes a bid $x_i \in \mathbb{R}_+$.[2] The utility $u_i(\omega, x)$ of player i, as a function of the state of nature ω and the bids $x = (x_1, ..., x_n)$, will be made precise below. A (pure) strategy of bidder i is a mapping $b_i : \Omega \to \mathbb{R}_+$, which is measurable w.r.t. Π_i, namely such that b_i is constant on every element of Π_i.

[2] The bidders will typically have different information partitions; they will thus be ex ante asymmetric. On the contrary, models with affiliated signals à la Milgrom and Weber (1982) are usually solved for ex ante symmetric bidders.

Let $\Pi_N = \bigvee_{j \in N} \Pi_j$ be the coarsest partition which refines all the Π_j's, $j \in N$. We assume that

$$\Pi_N = \{\{\omega\}, \omega \in \Omega\}, \text{ i.e., } \bigcap_{j \in N} \pi_j(\omega) = \{\omega\} \quad \text{for every } \omega \in \Omega$$

$$\text{and } v(\omega) = \omega \text{ for every } \omega \in \Omega \qquad \text{(assumption A)}$$

Under assumption A, Ω is a finite subset of $\mathbb{R}_+$ and a state of nature is interpreted as the bidders' best estimate of the value of the object when they share their information. In the sequel, we will not need to distinguish between this best estimate and the true value of the object[3].

We also assume, as in Einy et al. (2002), that the information partition Π_i of every bidder i, $i = 1, ..., n$, is *connected*; under assumption A, this means that, if ω_1, $\omega_2 \in \Pi_i$ and $\omega \in \Omega$ is such that $\omega_1 < \omega < \omega_2$, then $\omega \in \Pi_i$.

We can now make precise the utility functions $u_i : \Omega \times \mathbb{R}_+^N \to \mathbb{R}_+$, $i = 1, ..., n$:

$$u_i(\omega, x) = \frac{1}{g(x)}(\omega - \max_{j \in N \setminus i} x_j) I(x_i = \max_{j \in N} x_j)$$

where $g(x)$ denotes the number of winners (i.e., $g(x) = \mid \{i \in N : x_i = \max_{j \in N} x_j\} \mid$) and I is the indicator function. The auction game G is described as $G \equiv \left[N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{u_i\}_{i \in N}\right]$.

2.2 Sophisticated equilibria

Einy et al. (2002) prove that the original auction game G is dominance solvable; they refer to the corresponding solutions of G as "sophisticated equilibria". Among the sophisticated equilibria, they identify a particular equilibrium, in "$\beta-$strategies", which is computationally tractable. Under our assumption A, the "$\beta-$strategies" take the simple form

$$\beta_i : \Omega \to \mathbb{R}_+ : \beta_i(\omega) = \min \pi_i(\omega) = \min \{\omega' : \omega' \in \pi_i(\omega)\} \quad i \in N$$

The strategy β_i of player i thus consists of bidding the smallest possible value of the object, given his information. We observe that $\beta_i(\omega)$ does not

[3] Assumption A is not made in Einy et al. (2002) but basically amounts to a renaming of the states of nature. Such a renaming is innocuous as far as the full set of bidders N is fixed throughout the paper.

depend on the number of players. Einy et al. (2002) prove that $(\beta_i)_{i\in N}$ is a sophisticated equilibrium of G which interim Pareto-dominates all other sophisticated equilibria. The next lemma will be useful to compute expected payoffs at the equilibrium β.

Lemma 1

For every $\omega \in \Omega$, $\max_{k\in N}\beta_k(\omega) = \omega$. In particular, if $\beta_i(\omega) = \beta_j(\omega) = \max_{k\in N}\beta_k(\omega)$ for some $i \neq j$, then $\omega - \beta_i(\omega) = 0$. In other words, at the equilibrium β, ex aequo winning bids cannot generate a positive profit.

Proof: Let us set $\beta(\omega) = \max_{1\leq k\leq n}\beta_k(\omega)$. For every l, $1 \leq l \leq n$, by definition, $\omega \in \pi_l(\omega)$ and $\beta_l(\omega) = \min \pi_l(\omega)$; hence $\omega \geq \beta_l(\omega)$ for every l and $\omega \geq \beta(\omega)$. Let us show that, for every l, $\beta(\omega) \in \pi_l(\omega)$. This follows from the definitions if l is such that $\beta_l(\omega) = \beta(\omega)$; let thus consider l such that $\beta_l(\omega) < \beta(\omega)$; we still have $\beta_l(\omega) \in \pi_l(\omega)$ and $\omega \in \pi_l(\omega)$; on the other hand, $\beta_l(\omega) < \beta(\omega) \leq \omega$; since $\pi_l(\omega)$ is connected, we deduce $\beta(\omega) \in \pi_l(\omega)$. Hence $\beta(\omega) \in \cap_{1\leq j\leq n}\pi_j(\omega)$. By assumption A, $\beta(\omega) = \omega$. ■

From lemma 1, at the equilibrium β, the payoff of player i at ω is

$$\left(\omega - \max_{j\neq i}\beta_j(\omega)\right) I\left[\beta_i(\omega) > \max_{j\neq i}\beta_j(\omega)\right] \tag{1}$$

since ex aequos give a null contribution to the payoff. As a consequence of (1), player i's payoff at ω does not depend on possible rings among the other players. Player i only cares about the maximal bid of the others.

3 Auctions with collusion

3.1 Auxiliary game

A ring (or coalition) R is a subset of bidders. Let P be a coalition structure, namely a partition of N. Each element of P is interpreted as a bidding ring. For $R \subseteq N$, let $\Pi_R = \bigvee_{j\in R}\Pi_j$ be the coarsest partition which refines all the Π_j's, $j \in R$. Π_R describes the information of ring R if all its members share their information. From G and P, we construct an auxiliary game $G(P) \equiv$

$\left[P, (\Omega, p), \{\Pi_R\}_{R\in P}, \{U_R\}_{R\in P}\right]$, in which the players are the coalitions R, $R \in P$, the information partition of R is Π_R and the utility function of R is

$$U_R : \Omega \times \mathbb{R}^N_+ \to \mathbb{R}_+ : U_R(\omega, x) = \sum_{i\in R} u_i(\omega, x)$$

A (pure) strategy of R in $G(P)$ is a Π_R-measurable mapping $b_R : \Omega \to \mathbb{R}^R_+$. In $G(P)$, ring R is viewed as a single player, with information partition Π_R, who chooses a $R-$tuple of bids. The interpretation is that the members of coalition R share their information before jointly deciding on a bid profile $(x_i)_{i\in R}$. They make arbitrary transfers between each others. Information sharing is not submitted to incentive constraints if the bidders' information is verifiable (i.e., "hard"), an assumption which is often satisfied in the case of common values. If information is not verifiable, we show below (in lemma 3) that every ring R can implement any relevant strategy b_R by means of an incentive compatible mechanism.

3.2 Coalitional equilibria

As in Ray and Vohra (1997) and Ray (2007), we define a *coalitional equilibrium relative to* P as a Nash equilibrium $(b_R)_{R\in P}$ of $G(P)$. In $G(P)$, a strategy b_R of ring R is a profile of bids, one for every member of R. However, a strategy b_R such that, at some ω, several bids are positive would clearly be dominated. Hence, we henceforth describe a strategy of ring R as the single possibly positive bid of the ring. In particular, we define the strategy β_R of ring R as

$$\beta_R : \Omega \to \mathbb{R}_+ : \beta_R(\omega) = \min \pi_R(\omega) = \min \{\omega' : \omega' \in \pi_R(\omega)\}$$

where $\pi_R(\omega)$ is the element of $\Pi_R = \bigvee_{j\in R} \Pi_j$ which contains ω. The next lemma further characterizes the strategies β_R.

Lemma 2

For every $R \in P$ and *every* $\omega \in \Omega$, $\beta_R(\omega) = \max_{k\in R} \beta_k(\omega)$.

Proof: Let us assume that $R = \{1, 2\}$; the general case can be deduced by induction. Let us write 12 for $\{1, 2\}$. Without loss of generality, let us assume that $\beta_2(\omega) \geq \beta_1(\omega)$, namely that $\min \pi_2(\omega) \geq \min \pi_1(\omega)$. By connectedness and since $\pi_{12}(\omega) = \pi_1(\omega) \cap \pi_2(\omega) \neq \emptyset$, $\min \pi_2(\omega) \in \pi_1(\omega)$. Hence, $\beta_{12}(\omega) = \min \pi_{12}(\omega) = \beta_2(\omega) = \max \{\beta_1(\omega), \beta_2(\omega)\}$. ■

In order to implement the strategy β_R, ring R must achieve the information partition Π_R, which raises the issue of incentive compatibility if the information of R's members is "soft", i.e., not verifiable. More precisely, in the latter case, ring R must rely on a mechanism selecting bids and transfers as a function of *reports* that R's members make on their private information. We assume that player i's report must take the form of an estimate $e_i \in \mathbb{R}_+$ of the value of the object.

A *mechanism* $\mu_R \equiv (\tau_R, \alpha_R)$ for R consists of a bid function τ_R and an allocation rule α_R: given an $R-$tuple of reports $(e_i)_{i \in R}$, $\tau_R((e_i)_{i \in R}) \in \mathbb{R}_+$ is the single bid to be made by the ring and $\alpha_R((e_i)_{i \in R})$ determines the member of R who gets the object (and pays the auctioneer for it), together with balanced monetary transfers, if R wins the auction.

Given a mechanism $\mu_R \equiv (\tau_R, \alpha_R)$, we construct a revelation game between the members of R. This game takes place after that every member of R has committed to participate in R and starts with the choice of nature ω. Every member $i \in R$ learns the element $\pi_i(\omega)$ of his information partition which contains ω and then reports an estimate e_i to the mechanism. A strategy of player $i \in R$ in the revelation game is a Π_i-measurable mapping $\widehat{e}_i : \Omega \to \mathbb{R}_+$, which determines the estimate $\widehat{e}_i(\omega)$ that player i reports to the mechanism as a function of his information. Player i's payoff in the revelation game is determined by the allocation rule α_R. Player i's payoff thus depends on the estimates that are reported by the other members of R (indirectly through the bid that is made by R, which will make R win or not, and possibly directly through the allocation rule) but also on the bids that are made by the bidders outside the ring. The revelation game thus depends on the bidding strategies of the players who are not in the ring.

Let P be a coalition structure; we have seen that $\beta = (\beta_R)_{R \in P}$ is a coalitional equilibrium. For $R \in P$, the mechanism $\mu_R \equiv (\tau_R, \alpha_R)$ *implements* β_R *given* $(\beta_S)_{S \in P, S \neq R}$ if there exists an equilibrium $(\widehat{e}_i)_{i \in R}$ of the revelation game induced by μ_R and $(\beta_S)_{S \in P, S \neq R}$ such that for every $\omega \in \Omega$, $\tau_R((\widehat{e}_i(\omega))_{i \in R}) = \beta_R(\omega)$. The coalitional equilibrium $\beta = (\beta_R)_{R \in P}$ is *incentive compatible* if, for every $R \in P$, there exists a mechanism $\mu_R \equiv (\tau_R, \alpha_R)$ which implements β_R given $(\beta_S)_{S \in P, S \neq R}$.

Lemma 3

For every coalition structure P, the coalitional equilibrium $\beta = (\beta_R)_{R \in P}$ is incentive compatible.

Proof:

Let us fix P and $R \in P$. We must construct a mechanism $\mu_R \equiv (\tau_R, \alpha_R)$ which implements β_R given $(\beta_S)_{S\in P, S\neq R}$. Let the only significant bid of R be defined as $\tau_R((e_i)_{i\in R}) = \max_{i\in R} e_i$ for every $R-$tuple of reported estimates $(e_i)_{i\in R}$. Let the allocation rule α_R be defined as follows, independently of the reported estimates: if R wins the auction, then $i \in R$ gets the object with probability λ_i ($\lambda_i > 0$, $i = 1, ..., n$, $\sum_{i\in R} \lambda_i = 1$), in which case i also pays for the object[4]. Given this mechanism, the payoff of $i \in R$ in the revelation game is

$$\lambda_i \left(\omega - \max_{k\in N\setminus R} x_k\right) I\left[\max_{j\in R} e_j > \max_{k\in N\setminus R} x_k\right]$$

if ω is the state of nature, $(e_j)_{j\in R}$ is the vector of messages in R and $(x_k)_{k\in N\setminus R}$ is the vector of bids outside R.

We will show that $(\widehat{e}_j)_{j\in R} = (\beta_j)_{j\in R}$ is μ_R and $(\beta_S)_{S\in P, S\neq R}$ is an equilibrium of the revelation game induced by μ_R and $(\beta_S)_{S\in P, S\neq R}$, namely that players in $N \setminus R$ bid according to $(\beta_S)_{S\in P, S\neq R}$ and every member $j \neq i$ of R reports $e_j = \beta_j(\omega)$ to the mechanism, then $e_i = \beta_i(\omega)$ is a best response of player i.

From Einy et al. (2002) applied to $G(P)$ and lemma 2, $\beta_R = \max_{i\in R} \beta_i$ is a best response of ring R against $(\beta_S)_{S\in P, S\neq R}$. Applying once again lemma 2, $\beta_{N\setminus R} = \max_{S\in P, S\neq R} \beta_S$ and proceeding as in (1),

$$\begin{aligned} & E\left((\widetilde{\omega} - \beta_{N\setminus R}(\widetilde{\omega}))I\left[\beta_R(\widetilde{\omega}) > \beta_{N\setminus R}(\widetilde{\omega})\right] \mid \pi_R(\omega)\right) \\ \geq\ & E\left((\widetilde{\omega} - \beta_{N\setminus R}(\widetilde{\omega}))I\left[x_R > \beta_{N\setminus R}(\widetilde{\omega})\right] \mid \pi_R(\omega)\right) \end{aligned}$$

for every $\omega \in \Omega$ and $x_R \in \mathbb{R}_+$. If all members of R but player i report $e_j = \beta_j(\omega)$, the ring will not use its best response against $(\beta_S)_{S\in P, S\neq R}$ and player i, whose payoff is proportional to the ring's payoff, will possibly be harmed. The previous inequality holds in particular for $x_R = \max\left\{e_i, \beta_{R\setminus i}(\omega)\right\}$; since $\beta_{R\setminus i}$ is π_R-measurable, we can write

$$\begin{aligned} & E\left((\widetilde{\omega} - \beta_{N\setminus R}(\widetilde{\omega}))I\left[\beta_R(\widetilde{\omega}) > \beta_{N\setminus R}(\widetilde{\omega})\right] \mid \pi_R(\omega)\right) \\ \geq\ & E\left((\widetilde{\omega} - \beta_{N\setminus R}(\widetilde{\omega}))I\left[\max\left\{e_i, \beta_{R\setminus i}(\widetilde{\omega})\right\} > \beta_{N\setminus R}(\widetilde{\omega})\right] \mid \pi_R(\omega)\right) \end{aligned}$$

[4]Equivalently, if R wins the auction, every member i of R gets a share λ_i of R's total payoff.

By taking expectations w.r.t. $\pi_i(\omega)$, which is coarser than $\pi_R(\omega)$, we get

$$\begin{aligned} & E\left((\widetilde{\omega} - \beta_{N\setminus R}(\widetilde{\omega}))I\left[\beta_R(\widetilde{\omega}) > \beta_{N\setminus R}(\widetilde{\omega})\right] \mid \pi_i(\omega)\right) \\ \geq\ & E\left((\widetilde{\omega} - \beta_{N\setminus R}(\widetilde{\omega}))I\left[\max\left\{e_i, \beta_{R\setminus i}(\widetilde{\omega})\right\} > \beta_{N\setminus R}(\widetilde{\omega})\right] \mid \pi_i(\omega)\right) \end{aligned}$$

for every $\omega \in \Omega$ and $e_i \in \mathbb{R}_+$. By multiplying both sides of the latter inequality by λ_i, we conclude that player i cannot do better than $e_i = \beta_i(\omega)$.

There remains to check that for every $\omega \in \Omega$, $\tau_R((\widehat{e}_i(\omega))_{i\in R}) = \beta_R(\omega)$. By construction, $\tau_R((\widehat{e}_i(\omega))_{i\in R}) = \max_{i\in R}\beta_i(\omega) = \beta_R(\omega)$, where the last inequality follows from lemma 2. ■

Remark: If values are private and independent, an analog of lemma 3 holds for *any* coalitional equilibrium of any auction game (i.e., not necessarily second price) by relying on transfers à la Groves (1973) and d'Aspremont and Gérard-Varet (1979, 1982) (see Biran and Forges (2009)).

4 Core-stable rings

4.1 Cooperative game

The coalitional equilibria $(\beta_R)_{R\in P}$ derived in the previous section enable us to associate a cooperative game with the noncooperative game G.[5] For every coalition structure P and ring $R \in P$, let us define the worth $v(R;P)$ of R in P as the expected payoff of R at the equilibrium $(\beta_S)_{S\in P}$ of $G(P)$. The equilibrium payoff of coalition $R \in P$ at ω, which we denote as $v(R;P)(\omega)$, can be computed as in (1); using lemma 2,

$$\begin{aligned} v(R;P)(\omega) &= \left(\omega - \max_{j\in N\setminus R}\beta_j(\omega)\right) I\left[\max_{i\in R}\beta_i(\omega) > \max_{j\in N\setminus R}\beta_j(\omega)\right] \\ &= {}_{def}\psi(R)(\omega) \qquad (2) \end{aligned}$$

This expression shows that, for every ω, coalition R's payoff at ω does not depend on the coalition structure P, so that we can refer to it as $\psi(R)(\omega)$.

[5] We basically proceed as in Ray and Vohra (1997) and Ray (2007) even if, in our auction model, the uniqueness of coalitional equilibrium is not guaranteed. We rather focus on the equilibrium identified by Einy et al. (2002).

Coalition R's expected payoff does not depend on P either so that the partition form $v(R;P)$ reduces to a *characteristic function* ψ:

$$\begin{aligned} v(R;P) &= \psi(R) = E\left[\psi(R)(\widetilde{\omega})\right] \\ &= E\left[\left(\widetilde{\omega} - \max_{j\in N\setminus R}\beta_j(\widetilde{\omega})\right) I\left[\max_{i\in R}\beta_i(\widetilde{\omega}) > \max_{j\in N\setminus R}\beta_j(\widetilde{\omega})\right]\right] \end{aligned} \quad (3)$$

where $\widetilde{\omega}$ is the random variable representing the common value of the object.

The characteristic function ψ is similar to the one which has been derived for second price auctions with independent private values (see Mailath and Zemsky (1991) and Barbar and Forges (2007)). In order to see this, let $\omega \in \Omega$. If $\max_{i\in R}\beta_i(\omega) > \max_{j\in N\setminus R}\beta_j(\omega)$, then $\max_{i\in R}\beta_i(\omega) = \max_{i\in N}\beta_i(\omega) = \omega$. Using the notation $h^+ \equiv \max\{h, 0\}$ for any function h, we can thus write

$$\psi(R)(\omega) = \left(\max_{i\in R}\beta_i(\omega) - \max_{j\in N\setminus R}\beta_j(\omega)\right)^+ \quad (4)$$

and

$$\psi(R) = E\left[\left(\max_{i\in R}\beta_i(\widetilde{\omega}) - \max_{j\in N\setminus R}\beta_j(\widetilde{\omega})\right)^+\right] \quad (5)$$

By identifying $\beta_i(\widetilde{\omega})$ with the evaluation $\widetilde{v}_i$ of player i in an independent private value model, (5) is exactly the (first best) expected payoff of ring R in a second price auction at the equilibrium in dominant strategies (see Barbar and Forges (2007)). However, such a representation is not general at all, even in second price auctions. For instance, a partition form game (rather than a characteristic function) may be needed to account for the coalitions' interaction in models of common values which do not satisfy Einy et al. (2002)'s assumptions (see Barbar and Forges (2007), example 4)[6].

The following property of ψ will be useful.

Lemma 4

The characteristic function ψ defined by (3) is supermodular.

Proof: Recall (e.g., from Shapley (1971) or Moulin (1988)) that ψ is supermodular if for every coalitions R, S such that $S \subseteq R$ and every $k \in N \setminus R$

$$\psi(R \cup \{k\}) - \psi(R) \geq \psi(S \cup \{k\}) - \psi(S) \quad (6)$$

[6] An effective partition form game is also needed in first price auctions with independent private values (see Biran and Forges (2009)).

This inequality is easily checked case by case, at every $\omega \in \Omega$. Alternatively, using (4) and (5), ψ is basically a cost allocation game so that its supermodularity can de deduced from standard properties (see Littlechild and Owen (1973) or, e.g., Moulin (1988)).

More precisely, let $x_i \in \mathbb{R}_+$, $i \in N$ and $x(R) = \max_{i \in R} x_i$ for every $R \subseteq N$; x defines a standard cost allocation game and is thus submodular. Let f be the characteristic function defined by

$$\begin{aligned} f(R) &= (x(R) - x(N \setminus R))^+ \\ &= \left(\max_{i \in R} x_i - \max_{j \in N \setminus R} x_j \right)^+ \qquad \text{for every } R \subseteq N. \end{aligned}$$

The marginal contributions of f and x are related to each other by

$$f(R \cup \{k\}) - f(R) = x(N \setminus R) - x((N \setminus R) \setminus \{k\}) \quad \text{for every } R \subseteq N, k \notin R.$$

so that f is supermodular. The result holds in particular for $x_i = \beta_i(\omega)$, $i \in N$, and $f = \psi(\omega)$. ■

4.2 Core

Recall that the core of ψ is the set $C(\psi)$ of vector payoffs $(z_i)_{i \in N} \in \mathbb{R}^N$ such that $\sum_{i \in N} z_i = \psi(N) = E(\widetilde{\omega})$ and $\sum_{i \in R} z_i \geq \psi(R)$ for every ring $R \subseteq N$. $C(\psi)$ can be referred to as the *ex ante incentive compatible core* of the auction game G, i.e., as an analog to the solution concept previously defined for exchange economies by Forges and Minelli (2001) and Forges et al. (2002). Indeed, the coalitional equilibria behind the characteristic function ψ are incentive compatible by lemma 3 and ψ evaluates the worth of coalitions as their ex ante expected payoffs.

If $C(\psi)$ is not empty, the ring involving all the bidders can propose to share $\psi(N) = E(\widetilde{\omega})$ in such a way that all subrings $R \subseteq N$ agree to participate. We then say that N is *core-stable*.

Before showing that N is indeed core-stable in our model, let us summarize the commitment process behind our stability concept. Every ring R considers to form at the ex ante stage, i.e., before the choice of the state of nature ω. If R forms, every member $i \in R$ must commit to participate at that stage. R expects that the players $j \in N \setminus R$ will bid according to β_j but need not make conjectures on possible other rings. R will use the mechanism $\mu_R \equiv (\tau_R, \alpha_R)$ described in the proof of lemma 3, which implements β_R

given $(\beta_j)_{j\in N\setminus R}$, in particular allocate the object to $i \in R$ with some probability $\lambda_i^R > 0$ if R wins (with $\sum_{i\in R}\lambda_i^R = 1$). The mechanism guarantees that the members of R correctly report their estimates after having received their private information and that the sum of the ex ante expected payoffs of the members of R is $\psi(R)$. This scenario holds in particular for the grand coalition N. When N considers to form, N proposes a share z_i of $\psi(N)$ to every $i \in N$. The vector payoff $z = (z_i)_{i\in N}$ will induce ex ante participation of every subring R (which has chosen its mechanism μ_R leading to $\psi(R)$) if and only if $z \in C(\psi)$. In order to achieve z, N chooses the probabilities λ_i^N of its mechanism μ_N in an appropriate way.

The previous approach can be applied to test the stability of any specific ring R. Let ψ^R be the restriction of ψ to R, namely the characteristic function defined by $\psi^R(S) = \psi(S)$ for every $S \subseteq R$. We say that R is *core-stable* if $C(\psi^R)$ is not empty. Cooperation in R can be studied exactly as in N, since, as far as the β strategies are chosen, R does not care about the rings that might form outside R. As above, R chooses the probabilities λ_i^R of μ_R to guarantee a payoff $(z_i)_{i\in R}$ in $C(\psi^R)$.

Proposition

Let G be an auction game satisfying the assumptions of in section 2; all rings are core-stable in G.

Proof: By lemma 4, ψ is supermodular. (6) shows that, for every $R \subseteq N$, the same property must also hold for ψ^R. Using Shapley (1971), $C(\psi^R)$ is not empty. ■

5 Some examples

Example 1

$n = 3$, $\Omega = \{l, m, h\}$, $l < m < h$, $\Pr(\omega) = \frac{1}{3}\ \forall\omega$, $\Pi_1 = \{\{l\}\{m,h\}\}$, $\Pi_2 = \{\{l,m\},\{h\}\}$, $\Pi_3 = \{\{l,m,h\}\}$.

ω	$\beta_1 = \beta_{13}$	$\beta_2 = \beta_{23}$	β_3	β_{12}	$\psi_1 = \psi_{13}$	$\psi_2 = \psi_{23}$	ψ_3	ψ_{12}
l	l	l	l	l	0	0	0	0
m	m	l	l	m	$m-l$	0	0	$m-l$
h	m	h	l	h	0	$h-m$	0	$h-l$

$\psi_{123}(\omega) = \omega\ \forall\omega$.

Ex ante: $\psi_1 = \psi_{13} = \frac{1}{3}(m-l)$, $\psi_2 = \psi_{23} = \frac{1}{3}(h-m)$, $\psi_3 = 0$, $\psi_{12} = \frac{1}{3}(m-l) + \frac{1}{3}(h-l)$, $\psi_{123} = \frac{1}{3}(l+m+h)$.

Let $R = \{1, 2\}$. $\lambda_1 = \lambda_2 = \frac{1}{2}$ yields ex ante payoffs of $\frac{\psi_{12}}{2}$ to each member of R, which may not be individually rational: if e.g., $l = 1$, $m = 2$ and $h = 5$, $\psi_1 = \frac{1}{3}$, $\psi_2 = 1$ and $\psi_{12} = \frac{5}{3}$. Together with an ex ante transfer t, $\frac{1}{6} \leq t \leq \frac{1}{2}$, from player 1 to player 2, ex post equal sharing becomes individually rational. For instance, the Shapley value $Sh_1 = \frac{1}{2}$, $Sh_2 = \frac{7}{6}$ can be obtained in this way.

Leaving aside the incentive problem, the Shapley value can be used as a nice, ex post allocation rule. At $\omega = m$, player 1 would have won at the same price, without player 2, hence player 1 should get the whole surplus $m - l$. At $\omega = h$, the ring gets the object at price l, while, without the help of player 1, player 2 would have won it at price m; the surplus from cooperation is $m - l$, which can be shared by the two players: player 1 gets $\frac{m-l}{2}$ and player 2 gets $(h-m) + \frac{m-l}{2}$.

Remark: under assumption A, a player with superior information (see Einy et al. (2001)) is fully informed here. Assume player 1 is such a player. Then $\psi(R) = E(\widetilde{\omega})$ for every $R \ni 1$ and $\psi(R) = 0$ otherwise. The only payoff in the core gives $E(\widetilde{\omega})$ to player 1 and 0 to the others.

Example 2

$n = 3$, $\Omega = \{l, m, h\}$, $l < m < h$, $\Pr(\omega) = \frac{1}{3}\ \forall \omega$, $\Pi_1 = \{\{l\}\{m, h\}\}$, $\Pi_2 = \{\{l, m\}, \{h\}\}$, $\Pi_3 = \{\{l, h\}, \{m\}\}$

The information partitions of the players are not connected. Einy et al. (2001)'s results apply. The grand coalition is not stable ($\psi(R) = \psi(N)$ for $\mid R \mid = 2$).

Which practical conclusions can we draw from our analysis?

Einy et al (2002) model the bidders' information in a general way, except perhaps for the connectedness of the information partitions. This assumption is natural enough to hold in many relevant examples. If a bidder cannot distinguish beween two endpoints of some range of values, he will likely not be able to distinguish between the values in the entire range. In example 2, for instance, player 3's partition is not connected: his technology does not enable him to distinguish between a low and a high value, but enables him to assess whether the value is intermediate, which is rather odd. Another important feature of the model is of course the second pricing rule of underlying auction.

Under Einy et al (2002)'s assumptions, we consider an arbitrary ring, which forms exogenously (e.g., consisting of local producers in a procurement auctions) before the bidders receive their private information. We establish that the ring can design a budget balanced incentive compatible mechanism of collusion so that no subgroup of bidders can profit from seceding from the ring. Hence, in the absence of further specification of the auction rules, even large rings will be able to enforce collusive bidding.

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