Contracts with Aftermarkets - Hidden Action

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1 Introduction

It is well known that financial markets can favor the efficient allocation of resources to production and the sharing of risk. Less is known about their effects on incentives, for those who have limited access to them. In this paper I propose a stylized model of firms and financial markets, to capture these effects. The concern here is how different access to markets can affect the incentives to production.

This paper relates to two branches of economic theory. The literature on Walrasian economies in presence of Moral Hazard issues, and the literature on endogenous securities. I will discuss the fit of my work in these literatures, but first I am going to relate my work to two recent papers, which are particularly relevant to the discussion.

Parlour and Walden (2008) construct a model based on CAPM where workers are also the dispersed owners of the firms in the economy. Their assumptions and their main object of interest are opposite of mine. By assuming workers are the only traders, and cleverly constructing a simple shock that cannot go be traded away on markets, they mostly analyze the effects of moral hazard on financial markets. I am on the other hand more interested on the effect of financial markets on contracts. For this reason I cannot take firms as abstract risk neutral principals at the contracting stage. In fact Risk Neutrality corresponds only to a special case (firms with independent returns).

Magill and Quinzii (2005) ask a different although related questions. What set of securities is needed, for risk sharing and incentives to coexist? They study several cases, distinguished by the amount of information available for contracting, and for each case characterize the set of security which allow (constrained) efficiency. In this paper securities are determined in equilibrium, but they are chosen out of a very simple set. In the future it would be worth describing the endogenous choice of complex securities, to study the efficiency of financial engineering, in terms of incentives and risk sharing, but this is out of the scope of this paper.

There is by now a vast literature on moral hazard in general equilibrium settings. Helpman and Laffont (1975) and Prescott and Townsend (1984) are among the first to tackle the topic. Their works are concerned with efficiency in exchange and production economies but individuals can exert a costly, unobservable payoff relevant action. These papers are the first in a long, but not large, series of works, which extend the study of efficiency to more general economies. This work is different in that I include financial markets, and I trade off some generality for more precise comparative statics results.

A strand of the asset pricing literature looks at asset pricing in the presence of delegated portfolio management (for a survey, see Stracca, 2003). An example of the approach typical in these papers can be seen in Ou Yang's paper. These studies look at the effects on prices and returns of the classical informational asymmetries phenomena. Moral hazard and adverse selection are largely studied in a CAPM or APT setting, in which a representative principal delegates his investing decisions to an agent. In this literature inefficiencies take the form of deviations from the non-delegated case equilibrium. These deviations can take the form of changes in asset prices and optimal portfolio composition. Besides the different object of interest, the perspective in these works is in a sense opposite of the one taken here. There we have informed parties trading, whereas in the present work it is the uninformed parties accessing markets.

In Section 2 I present the model. In Section 3 I define the notion of equilibrium. In section 4 I show existence and uniqueness. In section 5 I analyze the effect of markets on production. Section 6 concludes. The appendix discusses issues of the Markowitz preferences, when used in a dynamic contest and some potentially unexpected effects of markets on contracts.

2 The Model

This stylized models is meant to capture certain features of interactions within firms and across firms in financial markets. It is helpful to introduce first the model of a single firm, and then proceed to define how they interact.

2.1 Primitives

A generic firm i is constituted by a principal-agent pair p_i, a_i. Each of these firms i generates random returns. All individuals have identical Mean Variance Preferences over random variables in the familiar form [1]

$$U(X) = \mu(X) - \frac{b}{2}\sigma^2(X)$$

Agents have a reservation utility. $\overline{u}$ Agents choose a costly action (which can be interpreted as the effort put into production) e_i from the interval $E = [\underline{e}, \overline{e}]$. The cost of effort is a continuous, strictly increasing real valued function $c(e)$.

The returns of a firm $\tilde{X}_i(e_i)$ depend stochastically on the effort level of the agent employed. Because of the assumptions on preferences, I can restrict my attention to mean and variance of firms returns, and express them as a function of all effort levels.

$$\mu_i(e_i)$$
$$\sigma^2(e_i)$$

A *contract* C_i is a contingent agreement on how to split returns between the Principal and the Agent forming firm i. I impose the restriction that these sharing rules be affine. If X is the random variable describing the profits of the firm, an admissible rule describing the principal's and agent's share must be of the form:

$$X_{a_i} = -\alpha + (1-\beta)X$$
$$X_{p_i} = \alpha + \beta X$$

[1] Note that all the analysis would apply to the CARA/normal framework, except that conditions for existence would be more general

with $\alpha \in \mathbf{R}$ and $\beta \in [0,1]$.

A contract C_i amounts to a pair α_i, β_i. In a binary setting this imposes the restriction that both shares be increasing in firm's returns.

Now consider that there are N firms. Let $e = [e_i(C_i)]_{1 \leq i \leq N}$ be the vector of efforts, and $C = [C_i]_{1 \leq i \leq N}$ be the vector of contracts in each firm .

The joint distribution of the vector of firm profits $\mathbf{X}(e) = [X_i(e_i)]_{1 \leq i \leq N}$ firms' profits will be uniquely determined by the effort levels across firms.

$$\mu(e) = \begin{bmatrix} \mu_1(e_1) \\ \vdots \\ \mu_N(e_N) \end{bmatrix}$$
$$\Omega_{jk}(e) = \rho_{jk}(e_j, e_k)\,\sigma_j(e_j)\,\sigma_k(e_k)$$

I will denote this vector and matrix as $\mu(e)$ and $\Omega(e)$.

Principals will have access to a financial market where they can trade their claims to profits and a riskless asset available in zero net supply.

The risky assets available can be described by a random vector, $\mathbf{X_p}(e, C) = [X_{p_i}(e_i, C_i)]_{1 \leq i \leq N} = [\alpha_i + \beta X_i(e_i)]_{1 \leq i \leq N}$

Because of our assumption on preferences, from now on I will simply identify the principals' shares with the vector of means $\mu(C,e) = \alpha + \beta\mu(e)$ and the variance-covariance matrix $\Omega(C,e) = \beta'\Omega(e)\beta$.

2.2 Timeline

I will consider here the case of "hidden action". In other words, the two parties cannot write contracts on the actual effort level. The conditions for existence trivially imply the conditions for existence when effort is observable.

CMN will denote the primitives of the game that are common knowledge at every stage, which are:

- The vector of means and the variance-covariance matrix of payoffs as a function of effort levels

$$\mu(\cdot)$$
$$\Omega(\cdot\cdot)$$

- The preferences of individuals (the variance aversion parameter b).
- The reservation utility of agents $\overline{u}$.

The economy reaches its equilibrium in 3 stages.

1. Each Principal p_i designs a contract $C_i = (\alpha_i, \beta_i)$. $\mathcal{C}_i$ is the set of all possible contracts for firm i. At this stage every principal knows CMN.

   The profile of offered contracts is $C = \{C_1, ..., C_N\} \in \mathcal{C} = \Pi_{i=1}^{N}\mathcal{C}_i$

2. Each agent a_i chooses his effort level, based on the contract C_i.

   A strategy of an agent in firm i is a function of contracts mapping to possible effort levels.

$$e_i : \mathcal{C}_i \rightarrow E$$
$$e_i : C_i \mapsto e_i\left(C_i\right)$$

   The strategy profile of all agents can be written as a vector of functions $e = \left[e_i\left(C_i\right)\right]_{1\leq i\leq N}$ of contracts offered.

3. *Before* uncertainty is realized, principals trade their claims to returns on an asset market, where a riskless asset L is available in zero supply. At this stage principals can observe all contracts and make conjectures on the level of effort and hence the distribution of returns of all firms. Every principal j will form some beliefs γ_i^j on the effort level e_i . $\gamma_i^j\left(e_i|C_i\right)$ is the cumulative distribution function induced by the beliefs on firm i under contract C_i . Principal j holds θ_j^i shares of firm i and the price of the stock of firm i is q_i

This table summarizes the choices each individual faces at a given time, and the information available to them.

When	**Who**	**What**	**Knowing What**
0	p_i	C_i	CMN
1	a_i	e_i	CMN, C_i
2	p_i	θ_i	CMN, C

Table 1: Timing

2.3 Payoffs

Let θ be the portfolio held by an agent. Let $\theta = \left(\theta_R|\theta_L\right)$ Where θ_R is an N-dimensional vector of positive holdings of the N risky assets, whereas θ_L is the position an investor holds in the riskless asset.

The ex-ante utility from a portfolio θ fixing the contracts C and effort choices e, is given by

$$\theta\cdot\left(\mu\left(C,e\right)|1\right) - \frac{b}{2}\theta_R'\mathbf{\Omega}\left(C,e\right)\theta_R$$

However principals don't observe the efforts e so they evaluate utility of portfolios based on contracts C and the beliefs they induce $\gamma\left(\cdot|C\right)$ [2]

[2]Note how the squared term is always positive. This will pose a problem, but in equilibrium beliefs will be degenerate and the term will be equal to zero

$$U_{p_i}(C,\theta,\gamma) =$$
$$\theta \cdot \int_E \mu(C,e|1)\,d\gamma(e|C) - \frac{b}{2}\int_E \left(\theta_R'\Omega(C,e)\,\theta_R + \left(\mu(C,e) - \int_E \mu(C,e)\right)^2\right) d\gamma(e|C)$$

Demand θ will depend on available assets and their prices (but prices are also a function of contracts).

Agents payoffs depend on the effort chosen and the contract in place.

$$U_{a_i}(C_i,e_i) = \mu_i(C_i,e_i) - \frac{b}{2}\sigma_i^2(C_i,e_i) - c(e_i)$$

3 Equilibrium

3.1 Description

Because individuals take their decisions at each stage looking at the final payoffs, Equilibrium is more easily described starting from the final stage of the game.

Asset Market Principals hold one unit of a security equal to their share of returns in their firm, and they all have the same information. Because all individuals have the same beliefs the solution concept used here is that of Arrow-Debreu Equilibrium. The equilibrium portfolios and prices will be based on expected payoffs induced by C. They will be a function $(\theta,q)(C,\gamma(e|C))$.

Contracting, the Agents' turn Each agent a_i observes the contract he is offered, C_i, and he knows his own type and the technology of the principal. This is all the payoff relevant information, so every agent is facing a choice between lotteries, and he is not playing against other players. They simply choose an effort level maximizing $U_{a_i}(\cdot)$. As noted their strategies will be functions $e_i(C_i)$.

Contracting, the Principals' turn Each principal designs a contract. They correctly conjecture the action of each agent, and the outcome of asset markets, given contracts. They can forecast the equilibrium path for all possible strategy profiles. Hence, this stage can be seen as a game principals play against each other. I will focus on pure strategies equilibria (and show their existence).

The flow of decisions is described schematically below, and the information available at each stage is summarized by the argument of the strategies.

$$C \to e(C) \to \theta(C, q(C), \gamma(e|C))$$

The utility in the first stage can be written as:

$$V_{p_i}(C) = U_{p_i}(C, \theta(\gamma(e|C)), \gamma(e|C))$$

3.2 Definition

An Equilibrium consists of

- A trading strategy θ_i^* for each Principal p_i and prices $q^* \in \mathbb{R}^N$ such that $[\theta^*, q^*](C, \gamma)$ is an Arrow-Debreu Equilibrium for the asset market when contracts are C . Each principal is endowed with one unit of one asset so that the endowment of principal p_i is $w_i = [0, 0, ..., 1, ..., 0, 0]$ with 1 being in the i-th position.

$$\begin{aligned}
&\theta_i^*(C, \gamma(e|C)) \in arg \max_{\theta_i \in \mathbb{R}_+^{N+1}} U_{p_i}(C, \theta_i, \gamma(e|C)) \\
&\text{such that} \\
&q^*(C, \gamma(e|C)) \cdot \theta_i(C, \gamma(e|C)) \leq q^*(C, \gamma(e|C)) \cdot w_i \\
&\sum_{i \in N} \theta_i^* = [\mathbf{1}_N | 0]
\end{aligned}$$

- Beliefs $\gamma^*(e|C)$ such that

$$supp(\gamma_i^*) \subseteq arg \max_{\tilde{e}_i \in E} U_{a_i}(C_i, \tilde{e}_i)$$

- For each agent a_i a strategy $e_i^*(C_i)$ such that

$$e_i(C_i) \in arg \max_{e_i \in E} U_{a_i}(C_i, e_i)$$

- For each principal p_i, a contract C_i^* such that

$$\begin{aligned}
C_i^* &\in arg \max_{C_i \in \mathcal{C}_i} V_i^*\left(C_i, C_{-i}^*\right) \\
&= U_{p_i}\left(\left(C_i, C_{-i}^*\right), \gamma^*\left(e|C_i, C_{-i}^*\right), \theta^*\left(\left(C_i, C_{-i}^*\right), \gamma^*\left(e|C_i, C_{-i}^*\right)\right)\right)
\end{aligned}$$

4 Existence of Equilibrium

4.1 Assumptions

4.1.1 Monotonic Preferences

It is well known that Mean-Variance Preferences need not monotonic. This could pose problems for the existence of equilibrium. In a standard CAPM setting, monotonicity of preferences is solved by imposing a bound on the variance aversion of every individual. Because I restrict attention to linear contracts, it is possible to show that, if preferences are monotonic for given returns, they will be monotonic for any prevailing contracts.

Definition 1. *Let X be a generic Random Variable on the state space $S = (s_1, ..., s_M)$ taking values $(x_1, ..., x_M)$. $U(X)$ is monotonic if $\frac{\partial U}{\partial x_i} > 0, \forall i$.*

Lemma 1. *Consider the preferences induced by the utility function*

$$U(X) = E(X) - \frac{b}{2} Var(X)$$

They are monotonic on a set of variables $\mathcal{X}$ defined on a finite state space S, if

$$b < \min_{X,s} \frac{1}{|x_s - \mu_X|}$$

Proof. The proof amounts to checking (by differentiating) under which conditions on b the utility function is increasing. $\square$

Lemma 2. *If preferences are monotonic for all feasible portfolios in an economy with assets characterized by returns (μ, Ω), then they will be monotonic on all feasible portfolios for any contracts $(\alpha_i, \beta_i)_{i\in N}$.*

Proof. For preferences to be monotonic for all feasible portfolios it has to be that

$$\begin{aligned} b <& \frac{1}{|\max \sum_{i\in N} \theta_i x_i - \sum_{i\in N} \theta_i \mu_i|} \\ =& \frac{1}{|\max \sum_{i\in N} \theta_i \left(x_i - \mu_i\right)|}. \end{aligned}$$

Where the max is taken across portfolios θ such that $\theta_i \in (0,1)$ and outcomes $x_i \in supp\,(X_i)$. Note that

$$\begin{aligned} & \sum_{i\in N} \theta_i \left[\left(-\alpha_i + (1-\beta_i)\right) x_i - \left[-\alpha_i + (1-\beta_i)\,\mu_i\right)\right] \\ =& \sum_{i\in N} \theta_i \left(\left(1-\beta_i\right) x_i - \left(1-\beta_i\right)\mu_i\right) \\ =& \sum_{i\in N} \theta_i \left(1-\beta_i\right)\left(x_i - \mu_i\right) \end{aligned}$$

I claim that

$$\max |\sum_{i\in N} \theta_i \left(1-\beta_i\right)\left(x_i-\mu_i\right)| \leq \max |\sum_{i\in N} \theta_i \left(x_i - \mu_i\right)|$$

Note that the solution to the maximization on both sides is going to be reached at the aggregate market portfolio so that the previous is equivalent to

$$\max |\sum_{i\in N} \left(1-\beta_i\right)\left(x_i-\mu_i\right)| \leq \max |\sum_{i\in N} \left(x_i - \mu_i\right)|$$

Since it will also be the case that at the maximum all the x_i's chosen will be greater (or smaller) than the μ_i's so that

$$\max \sum_{i\in N} \left(1-\beta_i\right)|\left(x_i-\mu_i\right)| \leq \max \sum_{i\in N} |\left(x_i - \mu_i\right)|$$

Observing that $\beta_i \in [0,1]$ concludes the proof $\square$

Assumption 1. *For a given set of assets, for every individual i, their risk tolerance parameter b_i lies on the interval $\left(0,\overline{b}\right)$, where $\overline{b} = \min_{X,s} \frac{1}{|x_s - \mu_X|}$*

This ensures that everyone's preferences will be monotonic.

4.1.2 Cost and Productivity of Effort

The purpose of the following assumptions is making sure that the only randomness in the economy is due to the uncertainty of firms' returns. Markowitz preferences are dynamically inconsistent and introducing further randomizations (such as some individual playing a mixed strategy) is undesirable in many ways. [3] Moreover, if only one effort choice is optimal for an agent, then principals can correctly infer the equilibrium effort by observing contracts. To achieve this I need agent's objective function to be concave in effort for any possible contract (α, β).

Assumption 2. *1. The cost function of an agent $c(e)$ is strictly increasing and strictly convex.*

$$\frac{\partial c}{\partial e} > 0, \frac{\partial^2 c}{\partial e^2} > 0$$

2. The effect of effort on the mean distribution of returns is such that utility is strictly increasing and concave in effort, the effect on the variance is strictly decreasing and convex for all $\left(0, \bar{b}\right)$

$$\frac{\partial \mu}{\partial e} > 0, \frac{\partial^2 \mu}{\partial e^2} \leq 0$$
$$\frac{\partial \sigma^2}{\partial e} < 0, \frac{\partial^2 \sigma^2}{\partial e^2} > 0$$

3. Moreover, I require the effect of effort on the variance to be bounded relative to the effect on mean returns.

$$\mu_e > |\sigma_e^2|$$
$$|\mu_{ee}| > |\sigma_{ee}^2|$$

Note how the first set of conditions implies that

$$\frac{\partial \mu_{X(e)}}{\partial e} - \frac{b}{2}\frac{\partial \sigma^2_{X(e)}}{\partial e} > 0$$
$$\frac{\partial^2 \mu_{X(e)}}{\partial e^2} - \frac{b}{2}\frac{\partial^2 \sigma^2_{X(e)}}{\partial e^2} < 0$$

Lemma 3. *Under Assumptions 1 and 2, the solution to the agent's problem, $e^*(\beta)$ is*

- *unique*
- *continuous*
- *decreasing*
- *concave*

[3] See Appendix

Proof. Note how $e^*(\alpha,\beta)$ solves

$$\max_e -\alpha + (1-\beta)\,\mu(e) - \frac{b}{2}(1-\beta)^2\,\sigma^2(e) - c(e)$$

The First Order Conditions of these problems amount to

$$(1-\beta)\,\mu_e(e) - \frac{b}{2}(1-\beta)^2\,\sigma_e^2(e) - c_e(e) = 0$$

Uniqueness. It follows from Assumption 2 that this derivative is a strictly decreasing function on $E = [\underline{e}, \overline{e}]$. To see this note that by, Assumption 2, concavity is guaranteed for any risk tolerance parameter in the interval $\left(0, \overline{b}\right)$. This implies that the assumptions for derivatives will also be true for

$$\mu_e - \frac{b}{2}(1-\beta)\,\sigma_e^2$$

for β between 0 and 1. This and the fact that cost is convex imply uniquness of the solution.

Continuity. for β in $[0,1)$ follows from the implicit function theorem. When $\beta = 1$ the optimal effort is $\underline{e}$.

Monotonicity and *Concavity.* Applying the implicit function theorem to the FOCs we obtain that

$$\begin{aligned}
\frac{\partial e^*}{\partial \beta} &= \frac{\mu_e - b(1-\beta)\,\sigma_e^2}{(1-\beta)\,\mu_{ee} - \frac{b}{2}(1-\beta)^2\,\sigma_{ee}^2 - c_{ee}} < 0 \\
\frac{\partial^2 e^*}{\partial \beta^2} &= \frac{\frac{b^2}{2}(1-\beta)^2\,\sigma_e^2\sigma_{ee}^2 + \mu_{ee}\mu_e - b(1-\beta)\,\sigma_{ee}^2\mu_e - bc_{ee}\sigma_e^2}{\left((1-\beta)\,\mu_{ee} - \frac{b}{2}(1-\beta)^2\,\sigma_{ee}^2 - c_{ee}\right)^2} < 0
\end{aligned}$$

□

Corollary 4. *In equilibrium, principals correctly conjecture the equilibrium effort of agents.*

$$\begin{aligned}
\gamma_i^*(e_i|C_i) &= 1, e_i \geq e_i^*(C_i) \\
&= 0, e_i < e_i^*(C_i)
\end{aligned}$$

This immediately from the definition of equilibrium and Lemma 3. By doing this, I am removing one potential layer of randomization. The objective function of a principal at the first stage will be.

$$U_{p_i}(C,\theta,\gamma^*) = \theta\cdot(\mu(C,e^*(C))\,|1) - \frac{b}{2}\theta_R'\mathbf{\Omega}(C,e^*(C))\,\theta_R$$

Lemma 5. *The best response of a principal is single valued.*

Proof. The proof amounts to showing that every principal is maximizing a strictly concave function on a convex set. Once the reaction function of the agent in incorporated in his individual rationality constraint, this does not delimit a convex region anymore. However, by substituting the

IR constraint in the objective function and Assumption 2.3, the resulting maximand is a strictly concave function on the interval E.

Let's start by studying the sign of the second derivative with respect to β of the objective function of a generic principal i. To make the proof more readable I abuse notation and suppress all the i subscripts.

$$\begin{aligned}
2e_\beta\left(\mu_e - 2\left(\frac{b}{N}-\frac{b}{2N^2}\right)\beta\sigma_e^2 - 2\left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)\sigma_e\right) &+ \\
e_\beta^2\left(\beta\mu_{ee} - \left(\frac{b}{N}-\frac{b}{2N^2}\right)\beta^2\sigma_{ee}^2 - \left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)\beta\sigma_{ee}\right) &+ \\
e_{\beta\beta}\left(\beta\mu_e - \left(\frac{b}{N}-\frac{b}{2N^2}\right)\beta^2\sigma_e^2 - \left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)\beta\sigma_e\right) &+ \\
-2\left(\frac{b}{N}-\frac{b}{2N^2}\right)\sigma^2 &
\end{aligned}$$

If $\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)$ is positive, it follows from Assumptions 2 and 1 that the addend on each line is negative. If the coefficient is negative we can observe that $\left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)$ is smaller than 1, by Assumption 1. This together with Assumption 2.3 implies that all the addends are negative as desired.

Let's now study the second derivative of the IR constraint reducing it to

$$\begin{aligned}
e_\beta^2\left[(1-\beta)\mu_{ee} - \frac{b}{2}(1-\beta)^2\sigma_{ee}^2 - c_{ee}\right] &+ \\
e_{\beta\beta}\left[(1-\beta)\mu_e - \frac{b}{2}(1-\beta)^2\sigma_e^2 - c_e\right] &+ \\
-2e_\beta\left[\mu_e - b(1-\beta)\sigma_e^2\right] &
\end{aligned}$$

By Lemma 3

$$e_\beta = \frac{\mu_e - b(1-\beta)\sigma_e^2}{(1-\beta)\mu_{ee} - \frac{b}{2}(1-\beta)^2\sigma_{ee}^2 - c_{ee}}$$

Substituting the first term with $e_\beta\left[\mu_e - b(1-\beta)\sigma_e^2\right]$

$$\begin{aligned}
e_{\beta\beta}\left[(1-\beta)\mu_e - \frac{b}{2}(1-\beta)^2\sigma_e^2 - c_e\right] &+ \\
-e_\beta\left[\mu_e - b(1-\beta)\sigma_e^2\right] &
\end{aligned}$$

The last term is problematic because it is positive.

Substituting for α in the objective function, we have $V(\beta)$ defined on $[0,1]$ The derivatives of this function will be given by

$$\frac{\partial^k V}{\partial \beta^k} = \frac{\partial^k U}{\partial \beta^k} + \frac{\partial^k IR}{\partial \beta^k}$$

Adding the last term of $\frac{\partial^2 IR}{\partial \beta^2}$ to the next to last term of $\frac{\partial^2 U}{\partial \beta^2}$

$$e_\beta \left(2 \left[\mu_e - 2 \left(\frac{b}{N} - \frac{b}{2N^2} \right) \beta \sigma_e^2 - 2 \left(\frac{b}{N} - \frac{b}{N^2} \right) \left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right) \sigma_e \right] - \left[\mu_e - b(1-\beta) \sigma_e^2 \right] \right) =$$
$$e_\beta \left(\mu_e - 4 \left[\left(\frac{1}{N} - \frac{1}{2N^2} \right) \beta + (1-\beta) \right] b \sigma_e^2 - 2 \left(\frac{b}{N} - \frac{b}{N^2} \right) \left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right) \sigma_e \right)$$

The term multiplying e_β needs to be positive. Inspecting the expression, it is clear that the "worse" possible scenario is when $\beta = 0$ and $\left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right)$ takes the lowest possible value (the largest negative value). As noted above $\left(\frac{b}{N} - \frac{b}{N^2} \right) \left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right)$ is smaller than 1.

A sufficient condition is for this expression to be always positive

$$\mu_e + b\sigma_e^2 + b\sigma_e > 0$$

But this is implied by Assumption 2.3 . The second derivative of V is hence negative, which concludes the proof. □

4.2 Existence

Theorem 6. *If the mean-variance preferences are monotonic for an asset market economy characterized by the mean vector μ and variance-covariance matrix Ω then there exists an equilibrium in the CAPM contracting economy.*

Proof. Because the Markowitz preferences are not expected utility -and they also fail to satisfy the 'betweenness axiom' (see Dekel, Safra, Segal (1991) -the best response correspondences of individuals would not be convexified by allowing for lotteries. This makes using the fixed point theorems by Glicksberg or Kakuthani impossible. I will instead show that the game can be reduced to a one-shot game in which principals have a *continuous* best response function *function* and apply Brouwer's fixed point theorem.

Principals' Turn: Asset Market. Principal utility will be given by the CAPM analytical solution.

$$U_{p_i} = q_i - \frac{\sum_{i \in N} q_i}{N} + \frac{\sum_{i \in N} \mu_i}{N} - \frac{b}{2} \frac{\mathbf{1'\Omega 1}}{N^2}$$

As noted earlier prices and returns q, μ, Ω are all continuous functions of contracts C and efforts e .

Agent's Turn: Effort Choice. By Lemma 3 effort level e are continuous functions of contracts C, hence the final utility of a principal can be described as a continuous function of contracts C .

Principals' Turn: Contract Design. By Lemma 5 Principals' problem is equivalent to a strictly concave optimization so that their best response function is always unique. By the maximum theorem it is also continuous.

The strategy space $\mathcal{C}$ is a rectangle, which is a convex, compact subset of $\mathbb{R}^{2N}$.

This satisfies the hypotheses of Brower fixed point theorem. There exist a fix point C^* which determines uniquely equilibrium efforts, beliefs, prices and portfolios. $C^*, e^*, \gamma^*, \theta^*, q^*$ form an equilibrium by construction.

□

5 The Insurance Effect of Markets - Moral Hazard

Consider the following special case. Individuals are identical in the sense that they all have Markowitz type of Mean-Variance preferences, and they all have the same risk tolerance coefficient. Technologies have different variances (denoted by σ_i^2 and different correlation coefficients ρ_{ij}). The mean returns of firms are determined by the effort of agents e_i, specifically $\mu_i(e_i) = e_i$. Cost is quadratic $c(e_i) = \frac{c}{2}e_i^2$

5.1 First Best Equilibrium

For the purpose of having a benchmark for optimal risk sharing and optimal effort, let's have a look at the first best (observable action) case.

5.1.1 No Markets

A Principal and an Agent agree on an action and a (random) payment.

$$
\begin{aligned}
&\max_{\alpha,\beta,e} \quad \alpha + \beta e - \frac{b}{2}\beta^2\sigma^2 \\
&\text{such that} \quad -\alpha + (1-\beta)\,e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e^2 \geq \overline{u} \qquad \text{(IR)}
\end{aligned}
$$

The optimal solution, action and contract is

$$
\begin{aligned}
e^* =& \frac{1}{c} \\
(\alpha^*, \beta^*) =& \left(-\frac{b}{8}\sigma^2 + \overline{u}, \frac{1}{2}\right)
\end{aligned}
$$

Proof. Substitute for α in the objective function to obtain

$$
\max_{\beta,e} \quad U(\beta, e) = (1-\beta)\,e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e_i^2 - \overline{u} + \beta e - \frac{b}{2}\beta^2\sigma^2
$$

which yield the first order conditions

$$\frac{\partial U}{\partial \beta} = a\,(1-\beta)\,\sigma^2 - a\beta\sigma^2 = 0$$
$$\frac{\partial U}{\partial e} = 1 - ce = 0$$

Which give the solution. $\square$

5.1.2 Financial Markets

Now I consider the case of many firms. to do this I solve the optimization problem of an arbitrary principal, who has now access to a financial market. The form of the utility function follows from similar considerations as in the hidden type case.

Lemma 7. *Principal i behaves as if his utility function were*

$$\begin{aligned} \alpha_i + \beta_i e_i + \\ + \left(\frac{b}{2N^2} - \frac{b}{N}\right)\beta_i^2\sigma_i^2 + \\ + \left(\frac{b}{N^2} - \frac{b}{N}\right)\sum_{j\neq i}\rho_{ij}\beta_i\beta_j\sigma_i\sigma_j \end{aligned}$$

Proof. Using the CAPM pricing formula, the riskless share of principal from firm i

$$\alpha_i + \beta_i\mu_i - \frac{b}{N}\left(\beta_i^2\sigma_i^2 + \sum_{l\neq i}\rho_{il}\beta_i\beta_l\sigma_i\sigma_l\right) +$$
$$-\frac{\sum_{j\in N}\alpha_j + \beta_j\mu_j}{N} + \frac{b}{N^2}\sum_{j\in N}\left(\beta_j^2\sigma_j^2 + \sum_{k\neq j}\rho_{jk}\beta_j\beta_k\sigma_j\sigma_k\right)$$

Each principal holds a fraction of the aggregate portfolio, in particular he holds a random variable from which he gets utility

$$\frac{\sum_{j\in N}\alpha_j + \beta_j\mu_j}{N} - \frac{b}{N^2}\left(\sum_{j\in N}\beta_j^2\sigma_j^2 + 2\sum_{k\neq j}\rho_{jk}\beta_j\beta_k\sigma_j\sigma_k\right)$$

Because utility is linear in mean, and the riskless asset has variance zero, adding the two and simplifying obtains the claim.

$\square$

$$\begin{aligned}
\max \quad & U_i^{MKT}(\alpha_i, \beta_i) = \\
& \alpha_i + \beta_i e_i + \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right) \beta_i^2 \sigma_i^2 + \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_i \beta_j \sigma_i \sigma_j \\
\text{s.t.} \quad & -\alpha_i + (1 - \beta_i) e_i - \frac{b}{2} (1 - \beta_i)^2 \sigma^2 - \frac{c}{2} e_i^2 \geq \overline{u} \qquad \text{(IR)}
\end{aligned}$$

The optimal solution, action and contract is

$$\begin{aligned}
e_i^{MKT} =& \frac{1}{c} \\
\beta_i^{MKT} =& \frac{\sigma_i^2 - \left(\frac{N-1}{N^2}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j}{\sigma_i^2 + \left(\frac{2N-1}{N^2}\right) \sigma_i^2}
\end{aligned}$$

Proof. Substitute for α_i in the objective function to obtain

$$\begin{aligned}
\max_{\beta_i, e_i} U_i^{MKT} & (1 - \beta_i) e_i - \frac{b}{2} (1 - \beta_i)^2 \sigma^2 - \frac{c}{2} e_i^2 - \overline{u} + \beta_i e_i + \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right) \beta_i^2 \sigma_i^2 + \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j
\end{aligned}$$

which yield the first order conditions

$$\begin{aligned}
\frac{\partial U_i^{MKT}}{\partial \beta} &= a (1 - \beta) \sigma_i^2 - \left(\frac{b}{N^2} - \frac{a2}{N}\right) \beta_i \sigma_i^2 - \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_i \beta_j \sigma_j = 0 \\
\frac{\partial U_i^{MKT}}{\partial e} &= 1 - ce = 0
\end{aligned}$$

Which give the solution. □

The key observation is that the optimal action stays the same, but the first best contract changes. As noted in the case of hidden type economies, the effect of markets is that principal act as if they were less risk averse. This changes the optimal risk sharing.

To better understand the effects of markets, let's focus on the special case in which all technologies are identical and so are their correlation coefficients.

$$\begin{aligned}
\sigma_i = \sigma_j &= \sigma, \forall i, j \\
\rho_{ij} &= \rho, \forall i, j
\end{aligned}$$

The symmetric solution in this case is

$$\beta = \frac{1 - \frac{(N-1)^2}{N^2}\rho\beta}{1 + \frac{2N-1}{N^2}}$$

So that in equilibrium

$$\beta^{MKT} = \frac{N^2}{N^2 + 2N - 1 + \rho\left(N^2 - 2N + 1\right)}$$

Note how, unless there is perfect correlation, in equilibrium principal always take more risk than in the no the market case $\left(\beta_i = \frac{1}{2}\right)$. In fact the equilibrium contracts will be identical to the no market case only if technologies are perfectly correlated $(\rho = 1)$.

$$\forall \rho < 1,$$
$$\beta^{MKT} < \frac{N^2}{N^2 + 2N - 1 + N^2 - 2N + 1} = \frac{1}{2}$$

5.2 Second Best Equilibrium

5.2.1 No Markets

Let's now turn to the more interesting case of unobservable actions. How do markets affect the equilibrium actions and returns? Here the decisions on risk sharing and effort are interdependent.

Again let's first look at a firm "in isolation"

$$\max_{\alpha,\beta,e} \quad \alpha + \beta e - \frac{b}{2}\beta^2\sigma^2$$
$$\text{such that} \quad -\alpha + (1-\beta)e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e^2 \geq \overline{u} \tag{IR}$$
$$e \in arg\max_{\tilde{e}} \quad -\alpha + (1-\beta)\tilde{e} - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}\tilde{e}^2 \tag{IC}$$

To simplify the problem let's start by solving the problem of an agent facing a given contract (α, β) ..

$$\max_{\tilde{e}} \quad -\alpha + (1-\beta)\tilde{e} - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}\tilde{e}^2$$

$$\max_{e} \quad -\alpha + (1-\beta)e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e^2$$

Because Individual Rationality can be optimally attained of with the transfer α the, optimal action e can be obtained from the first order condition of the agent problem.

$$(1-\beta)-ce=o$$

Plugging $e^* = \frac{1-\beta}{c}$ back into the Principal's objective function and the agent's IR constraint yields the following problem

$$\max_{\alpha,\beta} \quad \alpha+\beta\left(\frac{1-\beta}{c}\right)-\frac{b}{2}\beta^2\sigma^2$$

$$\text{such that} \quad -\alpha+\frac{(1-\beta)^2}{c}-\frac{b}{2}(1-\beta)^2\sigma^2-\frac{(1-\beta)^2}{2c}\geq \overline{u} \tag{IR}$$

The solution to this problem is

$$\beta = \frac{b\sigma^2}{2b\sigma^2+\frac{1}{c}}$$
$$e = \frac{bc\sigma^2+\frac{1}{c}}{2b\sigma^2+1}$$

Proof. Again substituting α results in

$$\max_{\alpha,\beta} \beta\left(\frac{1-\beta}{c}\right)-\frac{b}{2}\beta^2\sigma^2+\frac{(1-\beta)^2}{c}-\frac{b}{2}(1-\beta)^2\sigma^2-\frac{(1-\beta)^2}{2c}-\overline{u}$$

Differentiating with respect to β gives the first order condition

$$-2b\beta\sigma^2+b\sigma^2-\frac{\beta}{c}=0$$

The solution follows immediately from this and the fact that $e=\frac{1-\beta}{c}$ □

Note how $\beta < \frac{1}{2}$. Risk sharing is distorted to give the proper incentive to the agent. Note that $\beta > 0$, so that $e < \frac{1}{c}$. Because there is a trade-off between incentivizing to the agent and optimally sharing risk between two risk averse individuals the first best cannot be achieved.

I am now going to analyze how markets affect this tradeoff.

5.2.2 Financial Markets

Consider the market with many principals from the previous section, except now the action in each firm is unobservable.

$$\max \quad U_i^{MKT}(\alpha_i, \beta_i) =$$
$$\alpha_i + \beta_i e_i +$$
$$+ \left(\frac{b}{2N^2} - \frac{b}{N}\right) \beta_i^2 \sigma_i^2 +$$
$$+ \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j$$
$$\text{s.t.} \quad -\alpha_i + (1-\beta_i) e_i - \frac{b}{2}(1-\beta_i)^2 \sigma^2 - \frac{c}{2} e_i^2 \geq \overline{u} \tag{IR}$$
$$e_i \in arg \max_{\tilde{e}} \quad -\alpha + (1-\beta)\tilde{e} - \frac{b}{2} + (1-\beta)^2 \sigma^2 - \frac{c}{2}\tilde{e}^2 \tag{IC}$$

The solution is now

$$\beta_i^{MKT} = \frac{b\left(\sigma_i^2 - \left(\frac{N-1}{N^2}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j\right)}{b\left(\sigma_i^2 + \left(\frac{2N-1}{N^2}\right) \sigma_i^2\right) + \frac{1}{c}}$$

In the symmetric case

$$\beta^{MKT} = \frac{b\sigma^2 N^2}{b\sigma^2 \left(N^2 + 2N - 1 + \rho (N-1)^2\right) + \frac{N^2}{c}}$$

The solution differs from the first best case because of the $\frac{N^2}{c}$ term added to the denominator. So the equilibrium securities will be less risky than in the first best case. Moreover as observed for the non market case, the equilibrium effort is lower than at the optimum. Because β^{MKT} is again increasing in N, the equilibrium effort and hence returns are decreasing N.

In a fully symmetric case this result holds for every firm. This will not be the case if we drop the assumption of symmetry. However, the result will still hold in the aggregate in an economy where firms marginal distributions are identical, but not their conditionals. In other word we allow for different ρ_{ij} coefficients in the variance covariance matrix. This means that while every firm is identical when there are no markets, they are *ex-ante* different in terms of diversification opportunities they face (and offer). This can be interpreted as the existence of different "sectors".

Proposition 8. *Consider an economy with identical marginal distributions characterized by mean e_i, variance σ^2, cost of effort $\frac{c}{2}e^2$. The aggregate output with markets is lower than the aggregate output without markets.*

$$\Sigma_{i=1}^N e_i^* \geq \Sigma_{i=1}^N e_i^M$$

with the inequality holding strictly unless $\rho_{ij} = 1, \forall i, j$

Proof. Claim 1.

$$\Sigma_{i=1}^{N} e_i^* \geq \Sigma_{i=1}^{N} e_i^M \iff \Sigma_{i=1}^{N} \beta_i^* \leq \Sigma_{i=1}^{N} \beta_i^M$$

To see this note that

$$\begin{aligned}
\Sigma_{i=1}^{N} e_i^* \geq \Sigma_{i=1}^{N} e_i^M &\iff \\
\Sigma_{i=1}^{N} \frac{(1-\beta_i^*)}{c} \geq \Sigma_{i=1}^{N} \frac{\left(1-\beta_i^M\right)}{c} &\iff \\
\Sigma_{i=1}^{N} (1-\beta_i^*) \geq \Sigma_{i=1}^{N} \left(1-\beta_i^M\right) &\iff \\
\Sigma_{i=1}^{N} \beta_i^* \leq \Sigma_{i=1}^{N} \beta_i^M &
\end{aligned}$$

After estabilishing the above inequality. Observe that under the assumptions

$$\begin{aligned}
\beta_i^* &= \frac{1}{2 + \frac{1}{cb\sigma^2}} \\
\beta_i^M &= \frac{1 - \frac{N-1}{N^2} \Sigma_{j \neq i} \rho_{ij} \beta_j^M}{1 + \frac{1}{cb\sigma^2} + \frac{2n-1}{N^2}}
\end{aligned}$$

I need to prove that

$$\Sigma_{i=1}^{N} \frac{1}{2 + \frac{1}{cb\sigma^2}} \leq \Sigma_{i=1}^{N} \frac{1 - \frac{N-1}{N^2} \Sigma_{j \neq i} \rho_{ij} \beta_j^M}{1 + \frac{1}{cb\sigma^2} + \frac{2N-1}{N^2}}$$

Claim 2.

$$\begin{aligned}
\Sigma_{i=1}^{N} \beta_i^* \leq \Sigma_{i=1}^{N} \beta_i^M &\iff \\
\Sigma_{i=1}^{N} \beta_i^* \geq \Sigma_{i=1}^{N} \Sigma_{j \neq i} \frac{1}{N-1} \rho_{ij} \beta_j^M &
\end{aligned}$$

Note that the denominator of β_i^M can be rewritten as $2 + \frac{1}{cb\sigma^2} - \frac{(N-1)^2}{N^2}$. The first inequality amounts to

$$\begin{aligned}
\frac{N}{2 + \frac{1}{cb\sigma^2}} \leq \frac{N}{2 + \frac{1}{cb\sigma^2} - \left(\frac{N-1}{N}\right)^2} - \frac{\frac{N-1}{N^2} \Sigma_{i=1}^{N} \Sigma_{j \neq i} \rho_{ij} \beta_j^M}{2 + \frac{1}{cb\sigma^2} - \left(\frac{N-1}{N}\right)^2} &\iff \\
\frac{N(N-1)}{\left(2 + \frac{1}{cb\sigma^2} - \left(\frac{N-1}{N}\right)^2\right)\left(2 + \frac{1}{cb\sigma^2}\right)} - \frac{\frac{N-1}{N^2} \Sigma_{i=1}^{N} \Sigma_{j \neq i} \rho_{ij} \beta_j^M}{2 + \frac{1}{cb\sigma^2} - \left(\frac{N-1}{N}\right)^2} \geq 0 &\iff
\end{aligned}$$

Simplifying the last inequality

$$N(N-1)-\left(2+\frac{1}{cb\sigma^2}\right)\Sigma_{i=1}^{N}\Sigma_{j\neq i}\rho_{ij}\beta_j^M$$

Claim 2 immediately follows.

To conclude the proof, suppose by means of contradiction that the proposition did not hold, by Claim 1, $\Sigma_{i=1}^{N}\beta_i^* > \Sigma_{i=1}^{N}\beta_i^M$. By claim 2 this implies that $\Sigma_{i=1}^{N}\beta_i^* < \Sigma_{i=1}^{N}\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\beta_j^M$

Then it should be that

$$\Sigma_{i=1}^{N}\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\beta_j^M > \Sigma_{i=1}^{N}\beta_i^M$$

This can be rewritten as

$$\Sigma_{i=1}^{N}\left(1-\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\right)\beta_i^M < 0$$

which is impossible since $\rho_{ij} < 1, \forall i, j$

□

Financial Markets change the terms of risk sharing inside firms. Unlike the first best case, this has an effect on the equilibrium action, because it is exactly risk providing incentives for the agent to exert some positive effort. As the relative terms of risk sharing change, we move further away from the case in which the agent is risk neutral and optimal effort is obtained.

In this case, it seems markets reward firms for variance against returns, but that is not exactly the case. Markets reward firms for providing opportunities for hedging and diversification in specific states of the world.

To get a better intuition for this result, suppose there is a state space $S = \{SUN, RAIN\}$

Consider a principal owning the an ice cream factory. Her firm returns can take two values, $\overline{x}+e > \underline{x}+e$ respectively, in state SUN and in state $RAIN$. The security she is selling to the market will return $\alpha+\beta\overline{x}+\beta e$ and $\alpha+\beta\underline{x}+\beta e$. When designing an incentive contract she is playing with 3 variables α, β, e. When markets are not available, the underlying state representation does not matter, the solution α^*, β^*, e^* will be some tradeoff between risk sharing and incentives (increasing β decreases e and viceversa . However consider now what happens if there is another firm, whose securities return $\alpha'+\beta'\overline{x}+\beta e'$ when the state is $RAIN$ and $\alpha'+\beta'\underline{x}+\beta e'$ when the state is SUN. Because both principals are risk averse and there is now more return available in state $RAIN$, the owner of the ice cream factory will want to design a security generating more return in state SUN. Let's consider what happens to returns in each state when she increases the

3 components of her contract

$$
\begin{aligned}
&\Delta\alpha \\
RAIN &: \Delta\alpha \\
SUN &: \Delta\alpha \\
&\Delta\beta \\
RAIN &: \Delta\beta\underline{x} + \Delta\beta e \\
SUN &: \Delta\beta\overline{x} + \Delta\beta e \\
&\Delta e \\
RAIN &: \Delta e\beta \\
SUN &: \Delta e\beta
\end{aligned}
$$

Note how β is the only component of the contract which has a different effect in different states, and specifically it provides more exactly where needed: in state SUN. On the other hand, the agent, who does not access markets, values equally returns in equally likely states (risk aversion would make returns in $RAIN$ more valuable, if anything). Because markets change the relative value of returns in difference states for principals, the solution under markets will exhibit different risk sharing $\beta^{MKT} > \beta^*$ and consequently lower equilibrium effort $e^{MKT} < e^*$.

Understanding this effect allows to construct an example in which contracts and firms' output are affected in the opposite way as described above.

Consider an economy with two types of firms (50 of each kind). The low variance firms have variance .25, the high variance firms have variance 1. They are perfectly positively correlated $\rho_{ij} = 1, \forall i, j$ Risk aversion is .2, cost of effort is .5 . The average output without markets is given by $\frac{106}{56}$. Numerical evaluation yields an average output 2.11 .

In the appendix I discuss extensively how this particular distortion of risk sharing can occur even in a simple first best setting. The key intuition is again that the low variance firm has a comparative advantage in providing returns in the low state, where the other firms are obtaining an even lower performance.

6 Conclusion

This paper integrates a model of principal-agent interaction with asset markets. Each pair/firm produces random returns, whose distribution depends on agents' effort. Every Principal offers a contract to the Agent he is matched with, and the Agents make choose their costly action. What marks the difference from the standard contracting model is that Principals have access to an asset market on which they trade their shares of returns. This assumption is meant to capture the limited degree of access to financial markets, available to the average worker, who cannot entirely insure his labor risk.

I present a unified framework for which I define a notion of equilibrium and prove its existence and uniqueness. Under standard assumptions of contract theory, I study the interactions of financial markets on contracts.

On one hand, Moral Hazard inside firms induces suboptimal aggregate risk in an economy. On the other hand, introducing markets for principals, has ambiguous effects. If the marginal distributions of returns of firms are identical, markets will induce lower production levels. I construct an example, where asymmetry of marginal distributions and high degree of correlations induces riskier compensation packages in certain sectors and a positive effect on aggregate output.

Interesting directions for the future include making welfare comparisons with the case where workers can (at least partially) access financial markets, and making the decision of entering markets endogenous in a non trivial way (making access costly).

Introducing more complex model of firms (with many workers and more general contracts) would be technically non trivial, but an interesting way to study the effect of markets on multiplicity of equilibria (tackled experimentally by Kogan, Kwasnica and Weber) .

7 Appendix

7.1 Preferences

Markowitz preferences cannot be represented by expected utility. They cannot be represented by a linear functional in the space of mixtures. As a result they do not satisfy the independence axiom or -equivalently, but more significantly in this context- they are not "dynamically consistent".

Consider two random variables X_1 and X_2. We will have that

$$U(X_1) = E(X_1) - \frac{b}{2}Var(X_1)$$
$$U(X_2) = E(X_2) - \frac{b}{2}Var(X_2)$$

Now consider a mixture with probability p of X_1 and X_2. which we will call Y If individuals' preferences were "linear" we would have that.

$$U(Y) = pU(X_1) + (1-p)U(X_2)$$

Note that if a "linear" decision maker is indifferent between two random variables, he will also be indifferent between any mixture of them. Now compare with a Markowitz' type.

$$\begin{aligned} U(Y) > E(Y) - \frac{a}{2}Var(Y) = \\ pU(X_1) + (1-p)U(X_2) - \frac{a}{2}\left(pE\left(X_1 - E(Y)\right) + (1-p)\left(X_1 - E(Y)\right)\right) \end{aligned}$$

He would choose any of the two RV he is indifferent to above *any* mixture of the two. This implies that his best response in the space of mixed strategies is not convex[4]. This makes classical methods for proving existence (Berge's Maximum Theorem and Kakuthani's fixed point theorem) ineffective. To address this, I make sure that the model admits an equilbrium in the space of pure strategies.

As noted in the literature it is possible to construct pair of choices in between which some uncertainty is resolved, such that the agent makes plans he would not stick to. In particular in this model, this would lead to contracts which are not efficient as soon as they are signed even in a first best setting with symmetric information. Similar considerations would apply to the demand of assets on the security markets.

[4]Agents' preferences also fail to satisfy Between-ness as in Dekel (1986)

Here is an example of this issue. Consider a standard principal agent problem with two types of agents. The type of the agent is known as soon as principals and agents are matched

If the principal behaves linearly with respect to mixtures, his optimization problem looks like this

$$
\begin{aligned}
\max_{(\alpha_1,\beta_1,\alpha_2,\beta_2)} \rho & \left[\alpha_1 + \beta_1\mu_1 - \frac{a}{2}\beta_1^2\sigma_1^2\right] + \\
&(1-\rho)\left[\alpha_2 + \beta_2\mu_2 - \frac{a}{2}\beta_2^2\sigma_2^2\right] \\
s.t. \quad & IR_t : -\alpha_t + (1-\beta_t)\mu_t - \frac{a}{2}(1-\beta_t)^2\sigma_t^2 \geq \overline{u}
\end{aligned}
$$

The way principal actually treats mixture through mean variance preferences. The problem will look different because *the variance of a mixture is not the convex combination of variances.*

$$
\begin{aligned}
\max_{(\alpha_1,\beta_1,\alpha_2,\beta_2)} \rho & \left[\alpha_1 + \beta_1\mu_1 - \frac{a}{2}\left(\beta_1^2\sigma_1^2 + (\alpha_1 + \beta_1\mu_1)^2\right)\right] + \\
&(1-\rho)\left[\alpha_2 + \beta_2\mu_2 - \frac{a}{2}\left(\beta_2^2\sigma_2^2 + (\alpha_2 + \beta_2\mu_2)^2\right)\right] + \\
&\frac{a}{2}\left[\rho\left(\alpha_1 + \beta_1\mu_1\right) + (1-\rho)\left(\alpha_2 + \beta_2\mu_2\right)\right]^2 \\
s.t. \quad & IR_t : -\alpha_t + (1-\beta_t)\mu_t - \frac{a}{2}(1-\beta_t)^2\sigma_t^2 \geq \overline{u}
\end{aligned}
$$

$$
\begin{aligned}
\mu_1 &= 3\sigma_1^2 = 1 \\
\mu_2 &= 2\sigma_2^2 = \frac{1}{2} \\
\rho &= \frac{1}{2}
\end{aligned}
$$

The random portion of optimal contracts in the first case are given by $\beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2}$, which are exactly the same that the principal would offer to each type if he knew which type is the agent he was dealing with.

The Markowitz principal solution is instead $\beta_1 = .467\beta_2 = .539$, to which he would prefer $\frac{1}{2}, \frac{1}{2}$ as soon as uncertainty is resolved. Note that this Pareto improving renegotiation is not induced by voluntary information revelation from the agent, but simply by uncertainty resolving. Similarly, but more complicated, examples could be constructed for the asset market.

7.2 Riskier Contracts

Consider an economy, and for simplicity suppose that no action is needed for production to happen. In this economy the only role of contracting is for the principal and the agent to share risk.

The problem of a Principal who does not have access to markets is given by

$$\max_{\alpha,\beta} \quad \alpha + \beta\mu - \frac{b}{2}\beta^2\sigma^2$$

$$\text{such that} \quad -\alpha + (1-\beta)\mu - \frac{b}{2}(1-\beta)^2\sigma^2 \geq \overline{u} \tag{IR}$$

The optimal contract will induce symmetric risk sharing in equal parts.

Let's first see what happens in a small market. Consider two firms. Both firms will have mean returns μ. One firm has variance $\overline{\sigma}^2$ and the other has variance $\underline{\sigma}^2$, with $\overline{\sigma}^2 > \underline{\sigma}^2$.

Proposition 9. *There are values of $\overline{\sigma}, \underline{\sigma}, \rho$ defining two firms economies for which the low risk firm offers riskier contracts when markets are present.*

Proof. The problem of the Principal in the low risk firm.

$$\begin{aligned} \max \quad & U^{MKT}\left(\underline{\alpha}, \underline{\beta}\right) = \\ & \underline{\alpha} + \underline{\beta}\mu - \frac{3}{8}b\underline{\beta}^2\underline{\sigma}^2 - \frac{1}{4}b\rho\underline{\beta}\overline{\beta}\underline{\sigma}\overline{\sigma} \\ \text{s.t.} \quad & -\underline{\alpha} + \left(1-\underline{\beta}\right)\mu - \frac{b}{2}\left(1-\underline{\beta}\right)^2\underline{\sigma}^2 2 \geq \overline{u} \end{aligned} \tag{IR}$$

And the problem of the principal with the high risk firm.

$$\begin{aligned} \max \quad & U^{MKT}\left(\overline{\alpha}, \overline{\beta}\right) = \\ & \overline{\alpha} + \overline{\beta}\mu - \frac{3}{8}b\overline{\beta}^2\overline{\sigma}^2 - \frac{1}{4}b\rho\overline{\beta}\underline{\beta}\overline{\sigma}\underline{\sigma} \\ \text{s.t.} \quad & -\overline{\alpha} + \left(1-\overline{\beta}\right)\mu - \frac{b}{2}\left(1-\overline{\beta}\right)^2\overline{\sigma}^2 2 \geq \overline{u} \end{aligned} \tag{IR}$$

We can derive the optimal contracts given the other firm contracts.

$$\underline{\beta} = \frac{4\underline{\sigma} - \rho\overline{\beta}\overline{\sigma}}{7\underline{\sigma}}$$
$$\overline{\beta} = \frac{4\overline{\sigma} - \rho\underline{\beta}\underline{\sigma}}{7\overline{\sigma}}$$

And solve for equilibrium contracts.

$$\underline{\beta}^* = \frac{28\underline{\sigma} - 4\rho\overline{\sigma}}{49\underline{\sigma} + \rho^2\underline{\sigma}}$$
$$\overline{\beta}^* = \frac{28\overline{\sigma} - 4\rho\underline{\sigma}}{49\overline{\sigma} + \rho^2\overline{\sigma}}$$

Note how $\underline{\beta}^* < \frac{1}{2}$ if $\underline{\sigma} < \frac{2\rho}{7-\rho^2}\overline{\sigma}$. □

We have a case in which asset markets induce a riskier contract. Perhaps we can conclude that this is a problem of the small size of the market and. It will turn out it is not the case, and in fact large markets can even exacerbate this phenomenon.

Consider an economy with N firms. All firms have the same mean returns μ. A fraction γ of firms have variance $\overline{\sigma}^2$ and $1-\gamma$ have variance $\underline{\sigma}^2$, with $\overline{\sigma}^2 > \underline{\sigma}^2$.[5] . ρ controls the degree of correlation across firms. The difference in the variances controls how similar are the marginal distribution of returns for each firm.

Proposition 10. *There are values of $\gamma, \rho, \overline{\sigma}, \underline{\sigma}$ such that in a large economy low risk firms offer riskier contracts when markets are present.*

Proof. A generic Principal's problem

$$
\begin{aligned}
\max \ & U_i^{MKT}(\alpha_i, \beta_i) = \\
& \alpha_i + \beta_i \mu + \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right) \beta_i^2 \sigma_i^2 + \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_i \beta_j \sigma_i \sigma_j \\
\text{s.t.} \ & -\alpha_i + (1-\beta_i)\mu - \frac{b}{2}(1-\beta_i)^2 \sigma^2 \geq \overline{u} \qquad \text{(IR)}
\end{aligned}
$$

From which we get the following condition for optimal contracts

$$\beta_i^{MKT} = \frac{\sigma_i^2 - \left(\frac{N-1}{N^2}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j}{\sigma_i^2 + \left(\frac{2N-1}{N^2}\right) \sigma_i^2}, \forall i$$

Looking for a symmetric equilibrium we can write the optimal contracts as functions of other contracts for both types of firms.

$$
\begin{aligned}
\underline{\beta} &= \frac{\underline{\sigma}^2 - \frac{N-1}{N^2}\left[(N-1)(1-\gamma)\rho\underline{\beta}\underline{\sigma}^2 + (N-1)\gamma\rho\overline{\beta}\overline{\sigma}\underline{\sigma}\right]}{\underline{\sigma}^2 + \frac{2N-1}{N^2}\underline{\sigma}^2} \\
\overline{\beta} &= \frac{\overline{\sigma}^2 - \frac{N-1}{N^2}\left[(N-1)\gamma\rho\overline{\beta}\overline{\sigma}^2 + (N-1)(1-\gamma)\rho\underline{\beta}\underline{\sigma}\overline{\sigma}\right]}{\overline{\sigma}^2 + \frac{2N-1}{N^2}\overline{\sigma}^2}
\end{aligned}
$$

I am considering large economies so I will look at the contracts as $N \to \infty$

$$
\begin{aligned}
\underline{\beta} &= \frac{\underline{\sigma} - (1-\gamma)\rho\underline{\beta}\underline{\sigma} - \gamma\rho\overline{\beta}\overline{\sigma}}{\underline{\sigma}} \\
\overline{\beta} &= \frac{\overline{\sigma} - \gamma\rho\overline{\beta}\overline{\sigma} - (1-\gamma)\rho\underline{\beta}\underline{\sigma}}{\overline{\sigma}}
\end{aligned}
$$

[5] Of course γ should be rational and γN should be always an integer. This can be easily obtained by defining an initial economy by two integers g and h with g firms with high variance and h firms with low variance, and then replicating them. However, in this section I am interested in the behavior of large economies in which the two approaches yield the same results.

Solving the system we first obtain contracts for one type as a function of the other type.

$$\underline{\beta} = \frac{\underline{\sigma} - \gamma\rho\overline{\beta}\overline{\sigma}}{\underline{\sigma} + (1-\gamma)\,\rho\underline{\sigma}}$$
$$\overline{\beta} = \frac{\overline{\sigma} - (1-\gamma)\,\rho\underline{\beta}\underline{\sigma}}{\overline{\sigma} + \gamma\rho\overline{\sigma}}$$

And finally the equilibrium contracts

$$\underline{\beta} = \frac{\underline{\sigma} - \gamma\rho\,(\overline{\sigma} - \underline{\sigma})}{\underline{\sigma}\,(1+\rho)}$$
$$\overline{\beta} = \frac{\overline{\sigma} + (1-\gamma)\,\rho\,(\overline{\sigma} - \underline{\sigma})}{\overline{\sigma}\,(1+\rho)}$$

We can now observe that

$$\underline{\beta} < \frac{1}{2} \iff \underline{\sigma} < \frac{2\gamma\rho}{2\gamma + (1-\rho)}\overline{\sigma}$$

Note how this is never satisfied if $\rho = 0$ and always satisfied when $\rho = 1$ (since we assumed all along that $\overline{\sigma} > \underline{\sigma}$). □

An interesting fact is that in the limit, every worker gets exactly the same compensation package (ie: the same random variable).

Corollary 11. $-\underline{\alpha} + \left(1 - \underline{\beta}\right)\underline{X} = -\overline{\alpha} + \left(1 - \overline{\beta}\right)\overline{X}$

Proof.

$$\left(1 - \underline{\beta}\right)\underline{\sigma} = \frac{\rho\underline{\sigma} + \gamma\,(\overline{\sigma} - \underline{\sigma})}{1+\rho} = \frac{\rho\underline{\sigma} + \gamma\rho\,(\overline{\sigma} - \underline{\sigma})}{1+\rho}$$
$$\left(1 - \overline{\beta}\right)\overline{\sigma} = \frac{\rho\overline{\sigma} - (1-\gamma)\,(\overline{\sigma} - \underline{\sigma})}{1+\rho} = \frac{\rho\underline{\sigma} + \gamma\rho\,(\overline{\sigma} - \underline{\sigma})}{1+\rho}$$

We can conclude that agents obtain contract with the same standard deviation, and hence variance. Because agents are pushed to their IR constraint, if contracts have the same variance, they will also have the same mean. Because we are working with binary random variables it is enough to show that the compensation packages have the same mean and variance, to show that they are identical. □

By setting $\gamma = \frac{1}{2}$ we are constructing a replica economy of the initial 2 firms example. We can see that for certain parameters the contracts in the large economy will be even riskier than in the two firms case.

Proposition 12. *$\forall\rho > 1, \exists\overline{\sigma}, \underline{\sigma}$ such that*

$$\underline{\beta}\,(2) > \lim_{N\to\infty} \underline{\beta}\,(N)$$

Proof. The claim amounts to showing that there are variances such that

$$\frac{28\underline{\sigma} - 4\rho\overline{\sigma}}{49\underline{\sigma} + \rho^2\underline{\sigma}} > \frac{\underline{\sigma} - \gamma\rho\left(\overline{\sigma} - \underline{\sigma}\right)}{\underline{\sigma}\left(1+\rho\right)}$$

Some algebra shows that this boils down to

$$\underline{\sigma} < \left(\frac{\frac{1}{2}\rho^2 - 4\rho + \frac{41}{2}}{21 + \frac{1}{2}\rho^3 + \rho^2 - \frac{7}{2}\rho}\right)\rho\overline{\sigma}$$

The fraction on the right side is strictly positive, which allows us to conclude that we can find parameters satisfying our claim whenever $\rho > 0$. $\square$

How do markets (small and large) induce more "risk averse" behavior? A good intuition for this comes again from thinking in terms of two technologies with the same mean and different variances which are perfectly correlated ($\rho = 1$). In this case we can think of two states of the world L and H, with the returns of both types of firms being higher in H. The low variance firms have higher returns in L and lower in H than high variance firms do. Because of this they have an advantage in providing returns in state L. Markets make sure that this advantage is exploited giving incentives to firms to issue lower variance securities. A symmetric argument can be made for the high risk firms.

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