Asynchronous Revision Games with Deadline: Unique Equilibrium in Coordination Games

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Abstract

Two players prepare their actions before they play a normal-form coordination game at a predetermined deadline. In the preparation stage, each player stochastically obtains opportunities to revise their actions, and finally-revised action is played at the deadline. We show that, (i) A strictly Pareto-dominant Nash equilibrium, if there exists one, is the only equilibrium in the dynamic game; and (ii) in "battle of the sexes" games, (ii-a) the equilibrium payoff set is a full-dimensional subset of the feasible payoff set under perfectly symmetric payoff structure, but (ii-b) a unique equilibrium is selected with asymmetric payoff structure.

Keywords: Revision games, finite horizon, equilibrium selection, asynchronous moves

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1 Introduction

Many "once in a life time" problems involve coordination with other people. To name a few examples, two firms may need to decide which firm will conduct a particular task with very high fixed costs; A husband and a wife may want to choose their jobs at the same location; two researchers may want to decide which project to work on next, and so forth.

In such situations, that is, if the game is a high-stake event, it is often natural that players prepare their actions before the game is actually played, either because playing the game physically requires some preparation or because people would think about the game harder than about low-stake events.

The present paper models such situations. A two-player 2 x 2 game is played once and for all at a predetermined deadline, before which players have opportunities to revise their actions. These opportunities arrive stochastically and independently among both players. The stochastic component of the game reflects the fact that a game is typically not the only thing that players face. A player's hand may be tied for a while because she is playing another game. For example, a firm may be engaged in several tasks, and an employee in the firm may need to work for some other task, the necessary time for which is partially determined by the weather situation. In this context, we show that, (i) if there exists a Nash equilibrium that strictly Pareto-dominates all the other action profiles, then it is the only equilibrium; and (ii) in "battle of the sexes" games in which Nash equilibria are not Pareto ranked, (ii-a) while with perfectly symmetric payoff structure, the equilibrium payoff set is a full-dimensional subset of the feasible payoff set, (ii-b) a slight asymmetry is enough to select a unique equilibrium, which corresponds to the Nash equilibrium in the static game that gives the highest payoff to the "strong" player.

To model the preparation of actions, we use the framework of a finite horizon version of "revision games," proposed by Kamada and Kandori (2009). In revision games, players prepare their actions at opportunities in continuous time that arrive with a Poisson process until a predetermined deadline. The actions that are prepared most recently at the deadline are played once and for all. They consider the limit that the length of the preparation stage goes to infinity, so that there is enough number of opportunities by the deadline with a high probability. This is equivalent to saying that they consider the limit that the arrival rate becomes very high for a fixed length of preparation stage, that is, they consider the limit that players can revise their actions very frequently. This assumption is also maintained in the present paper. We also assume that the Poisson arrival of the revision opportunity is independent among two players.

Kamada and Kandori (2009) show that, with certain conditions such as continuous strategy space, non-Nash "cooperative" action profiles can be played at the deadline. As they have shown, this result would not be possible with a finite game with a unique pure Nash equilibrium.[1] Hence their focus is on expanding the set of equilibria when the static Nash equilibrium is inefficient relative to non-Nash profiles. We ask a very different question in this paper: we consider games with multiple efficient Nash equilibria, and ask which of these equilibria is selected.[2]

In the context of revision games with finite strategy sets, Kamada and Sugaya (2010) consider a model of election campaign with three possible actions (Left, Right, and Ambiguous). The main difference from their work is that they assume that once a candidate decides which of Left and Right to take, she cannot move away from that action. Thus the characterization of the equilibrium is substantially more difficult in the model of the present paper, because in our model an action a player has escaped from can be taken again by that player in the future.

The rough intuition for our results is as follows: First, consider a game with strictly Pareto-dominant Nash equilibrium. Firstly, we show that once both players prepare the strictly Pareto-dominant Nash equilibrium strategy, they will not escape from that state. Expecting that, if a unilateral change of the preparation by player i can induce the strictly

[1] The possibility of cooperation in finite horizon in Kamada and Kandori (2009) is closely related to that of finitely repeated games with multiple Nash equilibria (Benoit and Krishna, 1985).

[2] See also Ambrus and Lu (2009) for a variant of revision games model of bargainig in which the game ends when an offer is accepted.

Pareto-dominant Nash equilibrium, she will do so. In turn, player j whose unilateral change in the preparation can induce the situation where player i's unilateral change can induce the strictly Pareto-dominant Nash equilibrium, she will also do so if the deadline is far expecting that player i will go to strictly Pareto dominant Nash equilibrium.[3]

Second, consider the "battle of the sexes" game with symmetric payoffs. We prove that there are, in particular, three types of equilibria: The first one is such that the process sticks to the Nash profile that player 1 prefers, from the beggining until the end of the game. The second one is such that the process sticks to the Nash profile that player 2 prefers, again from the beggining until the end of the game. The most important is the third one, in which the process starts at an inefficient non-Nash profile that would give a player the best payoff if the opponent switches to another action. In this equilibrium, there is a "cutoff" time such that if a player obtains an opportunity after this cutoff while the opponent still sticks to the inefficient action, the player switches her action. Once the action profile reaches a Nash equilibrium, players will stick to that profile. Notice that, because of the symmetry of the payoff structure, two players' cutoffs must be the same, and the indifference condition at the cutoff implies that each player expects their respective "worse" Nash equilibrium payoff in this equilibrium.[4] We can then show that all the equilibrium payoff set is such that at each point in the set, at least one player gets the payoff of 1. Hence, the equilibrium payoff set is a full-dimensional subset of the feasible payoff set.

Finally, consider the "battle of the sexes" game with asymmetric payoffs. We prove that the only equilibrium payoff is the one that corresponds to the Nash equilibrium that the "strong" player (player 1) prefers, where by the "strong" player we mean the one who expects more in his preferred Nash equilibrium than the opponent (player 2) does in the

[3]Similar results are obtained in the literature. See Farrell and Saloner (1985) and Dutta (2003) for early works on this topic. Takahashi (2005) proves these results in a very general context.

[4]The intution is similar to the one of the "war of attrition." The war of attrition is analyzed in Abreu and Gul (2000) and Abreu and Pearce (2000). They consider the war of attrition problem in the context of bargaining. They show that if there is an "irrational type" with a positive probability, then the agreement delays in equilibrium because rational players try to imitate the irrational types. Players give in at the point where imitation is no longer profitable. Although the war of attrition is a common feature, the focus of their work and ours is quite different.

other pure Nash equilibrium. The reason is simple. Consider the "chicken race equilibrium," the third type of the equilibrium discussed in the case of the symmetric "battle of the sexes" game. Because we have asymmetry of payoffs now, the cutoffs of two players must differ. In particular, the strong player has to have a cutoff closer to the deadline than that of the weak player. This implies that, in the chicken race equilibrium, if it exists, the strong player expects strictly more than his "worse" Nash equilibrium payoff, while the weak player expects stricly less than her "worse" Nash equilibrium payoff. Hence, the strong player would not want to stick to the "worse" Nash equilibrium, which rules out the possibility of the second type of equilibrium in the symmetric case. Also, the weak player would not want to stick to the chicken race equilibrium, which rules out the third type of equilibrium in the symmetric case. Therefore the only possibility is the first type of equilibrium. Thus a unique profile is selected.

Our results crucially hinges on asynchronicity of the revision process. If the revision process were synchronous, the very same indeterminancy among multiple strict Nash equilibria would be present. The result that asynchronous moves select an equilibrium is not new, but the cases in which the asynchronous moves result in equilibrium selection is very limited. Lagunoff and Matsui (1997) show that in pure coordination games (games in which all players' payoff functions are the same) the Pareto efficient outcome is chosen, while Yoon (2001) shows that this result is nongeneric.[5] Our results, although in a bit different context in which the game is played only once at a deadline, show that the payoff structure can be anything in order to obtain the best outcome as a unique equilibrium.[6,7]

Let us compare our work with the large literature on equilibrium selection. The key

[5]Lagunoff and Matsui (2001) argue that this nongenericity result curucially depends on the order of the limits with respect to the discount factor and the purity of coordination.

[6]For asynchronicity with respect to the timing of signals in imperfect monitoring setting, see Fudenberg and Olszewski (2009).

[7]Caruana and Einav (2008) consider a similar setting as ours, with finite horizon and asynchronous moves, and show that the equilibrium in generic 2 x 2 game there is a unique equilibrium, irrespective of the order and timing of moves and the specification of the protocol. They assume that players incur a cost whenever they revise their actions, which substantially simplify their problem. In particular, on the equilibrium path, each player revises their actions at most once.

difference between our work and the past works in the literature is that, we consider a different class of situations than the ones considered in the literature. Specifically, four features distinguish our model from the existing literature on equilibrium selection. That is, we assume that (i) players are rational, (ii) they are nonanonymous, (iii) the structure of the game is common knowledge, and (iv) the game is played once and for all. Let us overview the related literature below: All of these four features seem to be present in the high-stake game examples mentioned in the very first paragraph of this Introdction.

Kandori, Mailath, and Rob (1993) and Young (1993) consider an evolutionary learning model in which players interact repeatedly, and each player's action at each period is stochastically perturbed. The key difference between their assumptions and ours is that in their model players are assumed to be boundedly rational –players are assumed to play myopically in repeated interactions, following a rule-of-thumbs type of decision rule, while we assume completely rational players. In addition, the game is repeated infinitely in their models, while the game is played once and for all in our model.

An evolutionary model called "perfect foresight dynamics" drops the bounded rationality assumption. Matsui and Matsuyama (1994) and Oyama, Takahashi, and Hofbauer (2008) consider a model in which patient players take into account the future path of the population dynamics. The key assumption to select an equilibrium is that there is a population of agents who are randomly and anonymously matched over time. In our model, on the other hand, fixed two players play a finite horizon revision game.

Global games are another successful way to select an equilibrium. Rubinstein (1981), Carlsson and van Damme (1993), Morris and Shin (1998), Sugaya and Takahashi (2009) show that non-existence of almost common knowledge due to the incomplete information can select an equilibrium. Without almost common knowledge, it is possible that a player thinks that his opponent thinks that the player thinks ... that the game is very different from the one that the player believes, which is the key to rule out risk-dominated equilibria. Thus, in particular, it is impossible to have a common knowledge that the game is in a small open neighborhood of some a priori specified one.

Lipman and Wang (2000) consider a finite horizon repeated games with small switching costs, in which the per-period length (henth the per-period payoff) is very small relative to the costs. They too use backward induction argument to select a unique equilibrium in games that we consider in the present paper. However, there are three important differences. First, as the frequency of revision increases, it is potentially possible in our model that players change their actions very often, while in Lipman and Wang (2000), the switching cost essentially precludes this possibility. Second, the game is repeatedly played in their model, while it is played only once at the deadline in our model. Hence, in particular, their model cannot have "chicken race equilibrium" type of strategies in equilibrium. Third, in their model, the prediction of the game is not robust to affine transformation of payoffs.[8] Hence, in some payoff specifications, their selection matches ours, while in other cases it does not. The reason for this scale-dependence is that the switching cost is directly payoff-relevant. In our case, detailed specifications of the preparation stage (such as the arrival rate) is not directly payoff-relevant, so our result is robust to the affine transformation of the payoffs.

All of these four lines of the literature result in a selection of risk-dominant equilibrium of Harsanyi and Selten (1988) in 2 x 2 games.[9] In our model, however, a different answer is obtained: a Pareto-dominant strict Nash equilibrium is played even if it is risk-dominated. Roughly speaking, since we assume complete information with nonanonymous players, there is no "risk" of mis-coordination, which ensures that rational players select a Pareto-dominant equilibrium.

[8]This nonrobustness is fine for their purpose since they focus on pointing out that the set of outcomes discoutinuously changes by the introdction of a small switching cost. However, it would not be very ideal in our context because we focus on identifying *which* outcome is selected by the introduction of the preparation stage. For that purpose, we want to minimize the information necessary for the selection: the preference of each player is sufficient information to identify the outcome and the information to compare different players' payoffs is not necessary.

[9]As an exception, Young (1998) shows that in the context of contracting, his evolutionary model does not necessarily lead to risk-dominant equilibrium (p-dominant equilibrium in Morris, Rob and Shin (1995)). But he considers a large anonymous population of players and repeated interaction, so the context he focuses on is very different from the one we focus on.

2 Model

We consider 2 x 2 normal-form games, i.e., games with two players and two pure actions for each player. The game π is described as follows:

	L	R
U	$\pi_1(U,L), \pi_2(U,L)$	$\pi_1(U,R), \pi_2(U,R)$
D	$\pi_1(U,D), \pi_2(U,D)$	$\pi_1(D,R), \pi_2(D,R)$

Before players actually take actions, they need to "prepare" their actions. We model this situation as in Kamada and Kandori (2009): time is continuous, $-t \in [-T, 0]$ with $T = 1$, and the normal form game (referred to as a "component game") is played once and for all at time 0. The game proceeds as follows. First, at time $-t = -1$, two players simultaneously choose actions. Between time $-T = -1$ and 0, each player independently obtains opportunities to revise their prepared action according to a Poisson process with arrival rate λ. At $t = 0$, the action profile that has been prepared most recently by each player is actually taken and each player receives the payoff that corresponds to the payoff specification of the component game. Each player has perfect informaion about past events at any moment of time. No discounting is assumed, although this assumption does not change any of our results.[10]

We consider the limit of the set of subgame-perfect equilibrium payoffs of this game as the arrival rate λ goes to infinity.

Let $\phi^\lambda(\pi)$ be the set of subgame-perfect equilibrium payoffs given arrival rate λ.

Definition 1. A payoff set $S \subset \mathbb{R}$ is a **revision equilibrium payoff set** of π if $S = \phi(\pi) := \lim_{\lambda\to\infty} \phi^\lambda(\pi)$. If $\phi(\pi)$ is a singleton, we say its element is a **revision equilibrium payoff**. The set of action profiles that correpond to the revision equilibrium payoff set is a **revision equilibrium set**. If the revision equilibrium set is a singlton, we say its element

[10]Discounting only scales down the payoff at time 0.

is a **revision equilibrium**.

That is, a revision equilibrium payoff set is the set of payoffs achievable by the revision game defined in this section. It will turn out in the sequel that this set is often a singleton. The term “revision equilibrium payoff” is used for a convenient abbreviation that represents the element of such a singleton set. “Revision equilibrium set” and “revision equilibrium” are analogously defined.

Note well that whenever we refer to some action profile (resp. a payoff profile) as a revision equilibrium (resp. a revision equilimrium payoff), we implicitly mean that it is the unique element of the revision equilibrium set (resp. revision equilibrium payoff set).

3 Pareto-Dominant Strict Nash Equilibrium in a Component Game

In this section we consider a component game with a Nash equilibrium that Pareto-dominates all other action profiles. Note that this condition is stronger than Pareto-ranked Nash equilibria. We will show that this strategy profile is selected in our model.

Proposition 1. *Suppose that* $\pi_i(U, L) > \pi_i(U, R), \pi_i(D, L), \pi_i(D, R)$ *for each* $i = 1, 2$. *Then,* (U, L) *is a unique revision equilibrium.*

This result is actually more general beyond the finite component games: In the Appendix, we prove this result in terms of an arbitrary strategy set X_i for each player i, as opposed to assuming only 2 actions for each player.

Let us sketch the proof method. Since fixing T and letting λ converge to infinity is equivalent to fixing λ and letting T converge to infinity, we consider the latter formulation for the sake of “backward induction.” First we show that once the action profile reaches the Pareto-dominant profile, then the action profile cannot escape from it from then on. To show this, we make a backward induction argument: At period $-\delta$ with sufficiently small

$\delta > 0$, no player can gain by deviating from the profile because the physically possible payoff change during the time between period $-\delta$ and 0 is very close to zero. Knowing this, at period -2δ, no player would deviate from the profile. And so on. Finally, knowing that the profile is an absorbing state, players would want to go to this profile if there is left much time to revise the actions, because the profile is better than any other profile for each player. Knowing this, each player expects that at time $-T$ the opponent plays only that profile.

Notice that the above equilibrium selection always selects the Pareto-dominant equilibrium in finite component games. In particular, even if the better Nash equilibrium is risk-dominated by a worse Nash equilibrium, the asynchronous revision process leads to a better equilibrium. The key is that, if the remaining time is sufficiently long, since it is almost common knowledge that the opponent will move to the Pareto-dominant equilibrium afterwards, the risk of mis-coordination can be arbitrarily small.

Notice also that we do not require that the game is of "pure-coordination," in which all players' payoff functions are the same. This result is in a stark difference from Lagunoff and Matsui (1997), in which they need to require that the game is of pure coordination, as otherwise their result would not hold (Yoon, 2001).

4 Pareto-Unranked Nash Equilibria in a Component Game: Battle of the Sexes

The result in the previous section suggests that the Pareto-dominant equilibrium is selected in games in which such a profile exists. But what happens if there are mutiple Pareto-optimal Nash equilibria? In this section we consider the game of "battle of the sexes," as follows:

$$g(\epsilon) = \begin{array}{ccc} & L & R \\ U & a+\epsilon, 1 & 0, 0 \\ D & 0, 0 & 1, a \end{array},$$

with $a > 1$ and $\epsilon \geq 0$. Observe that there are two pure strategy Nash equilibrium, (U, L) and (D, R), where player 1 prefers the former equilibrium while player 2 prefers the latter.

If $\epsilon = 0$, then two players are perfectly symmetric. Then it is obvious that the set of equilibrium payoff is symmetric. In Subsection 4.1, we show that this set is a full-dimensional subset of the feasible payoff set. This set for the case with $a = 2$ is depicted in Figure 1.[11] However, if ϵ is strictly positive, then what would happen is no longer obvious. We show in Subsection 4.2 that (U, L), which corresponds to the payoff $(a+\epsilon, 1)$, is a revision equilibrium. The revision equilibrium payoff in this case with $a = 2$ is depicted in Figure 2.

4.1 Anti-Equilibrium Selection in Symmetric Battle of the Sexes Game

In this subsection we analyze the case with $\epsilon = 0$.

Proposition 2. *Suppose that $\epsilon = 0$ in game g. Then, the revision equilibrium payoff set is $\phi(g(0)) = \{(y_1, y_2) \in [1, a]^2 \, | y_1 + y_2 \leq a + 1\}$.*

Figure 1 depicts the set $\phi(g(0))$, for the case of $a = 2$. Let us first discuss how to obtain the extreme points of the sets, assuming $a = 2$ for the sake of concreteness. First, point $(2, 1)$ can be obtained by just sticking to (U, L) on the path of the play. In a perfectly symmetric way, $(1, 2)$ can be obtained by just sticking to (D, R) on the path of the play. To obtain the payoff profile $(1, 1)$, we construct an equilibrium as follows: When $t > t^*$ for some appropriately chosen t^*, players 1 and 2 stick to U and R, respectively. For time $t < t^*$, if a player gets an opportunity, he gives up and change his action (to D if player 1 gives up and to R if player 2 gives up). From now on we refer to this equilibrium by "chicken race equilibrium," as the equilibrium has a favor of the "chicken race" game," in which two drivers drive their cars towards each other until one of them gives in, while if both do not give in then the cars crash and the drivers die. Two features of this equilibrium are important:

[11] All figures are placed at the end of this paper.

First, because two players are symmetric, their "cutoffs," t^*, must be the same. Second, at this t^*, they are indifferent between sticking to the current action and changing it, which gives him the payoff of 1. Hence in this equilibrium each player expects the payoff of 1. This is why the payoff profile $(1, 1)$ can be obtained.

Now we explain how to obtain the payoffs other than the extreme points of the equilibrium payoff set defined in Proposition 2. Because we are not assuming any public randomization device, it is not immediately obvious that these payoffs can be achieved. However, we can use the Poisson arrivals when t is large as if it were a public randomization device: players start their play at (U, R), and "count" the numbers of Poisson arrivals to each player and then decide which of three equilibria to play. Since the length of the time interval during which players count the arrivals can be arbitrary rational and irrational numbers, we can sustain all the payoffs in the equilibrium payoff set defined in Proposition 2.

4.2 Equilibrium Selection in Asymmetric Battle of the Sexes Game

In this subsection we consider the case with $\varepsilon > 0$.

Proposition 3. *Suppose that $\epsilon > 0$ in game g. Then, (U, L) is a unique revision equilibrium.*

The proposition says that the revision equilibrium payoff set in asymmetric battle of the sexes is a singleton. Figure 2 depicts the revision equilibrium payoff, for the case of $a = 2$. The intuition is as follows: First consider the symmetric case analyzed in the previous subsection. As noted, the cutoffs of the two players in the chicken race equilibrium are the same in this case. Now make ϵ strictly positive. Since player 1 becomes more patient than player 2 because player 1 expects more by player 2's giving up than player 2 does by player 1's giving up. This implies that the cutoffs are no longer the same. Specifically, player 1's cutoff needs to be closer to the deadline than player 2's cutoff does. But then, player 1 expects the payoff strictly more than 1 at the profile (U, R) when t is large, while player 2 expects strictly less than 1. This in particular has two implications: First, player 1 would be better off by deviating from the profile (D, R) when t is large. Second, player 2 would be better off by deviating from the profile (U, R) even when t is large. Hence the process must settle into the profile (U, L). There is more complication that arise when t is large, but the basic intuition for the result is as above.[12]

5 Concluding Remarks

We analyzed the situation in which two players prepare their actions before they play a normal-form coordination game at a predetermined date. In the preparation stage, each player stochastically obtains opportunities to revise their actions, and finally-revised action is played at the deadline. We showed that, (i) If there exists a Nash equilibrium that strictly Pareto-dominates all the other action profiles, then it is the only equilibrium; and (ii) in "battle of the sexes" games in which Nash equilibria are not Pareto ranked, (ii-a) while with perfectly symmetric payoff structure, the equilibrium set is a full-dimensional subset of the feasible payoff set, (ii-b) a slight asymmetry is enough to select a unique equilibrium, which corresponds to the Nash equilibrium in the static game that gives the highest payoff to the "strong" player.

Let us mention possible diections of future research. First, our analysis has been restriced to 2 x 2 games, but the basic intuition seems to extend to more general cases.[13] Second, it would be interesting to consider the case in which there exists only one mixed equilibrium. For example, in a symmetric "matching pennies" game, it is obvious that the probability distribution over the outcome is the same as in the mixed strategy equilibrium of the component game. A question is whether this is true for asymmetric case. Third, it would be interesting to see the hybrid version of synchronized and asynchronized revision games. These possibilities are out of the scope of this paper, but we believe that the present paper laid out motivations that are enough to pursue these generalizations.

[12] If t is large, player 2 would want to deviate from (D, R) to $(D, L,)$.

[13] We actually have not characterized equilibria for all payoff structure in 2 x 2 games. We are currently working on this, and we will include results for other class of 2 x 2 games in a revised version of this paper.

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A Appendix: Proof of Propositions

Since fixing T and letting λ converge to infinity is equivalent to fixing λ and letting T converge to infinity, we consider the latter formulation for the sake of "backward induction." We focus on right-continuous strategy. The following notations are useful: let σ be a strategy in the revision game. As we will see, σ only depends on the calendar time t and the action profile most recently prepared by the player. Therefore, letting x_i be the action most recently prepared by player i at t, we can write $\sigma_i^t(x) \in A_i$.[14] Finally, let $BR_i(x_{-i})$ be a static best response to x_{-i} by player i in the component game.

A.1 Proof of Proposition 4

We prove the result in much more general context. Consider a 2-player normal-form game $\pi = (\pi_1, \pi_2)$ with strategy set X_i for each player i.

First we define a terminology needed to state the result:

Definition 2. A strategy profile x^* is said to be **strongly strictly Pareto -dominant strategy profile** if there exists $\epsilon > 0$ such that for all i and all $x \in X_1 \times X_2$ with $x \neq x^*$, $\pi_i(x_i^*) > \pi_i(x) + \epsilon$.

Notice that in the above definition, the starategy sets can be either finite or infinite. In the case of finite strategy sets, the definition reduces to the standard definition of the game with strictly Pareto-dominant strategies.

We can prove the following result:

Proposition 4. *Suppose that x^* is strongly strictly Pareto-dominant strategy profile and* $\inf \pi_i =: \underline{\pi}_i > -\infty$*. Then, x^* is a unique revision equilibrium.*

Proof. To prove this result, we first verify that the following lemma holds:

[14] We can concentrate on the pure strategies without loss of generality.

Lemma 1. *Suppose that x^* is strongly strictly Pareto-dominant strategy profile and* $\inf \pi_i =: \underline{\pi}_i > -\infty$*. Then if a subgame starts with a profile x^*, there is a unique subgame perfect equilibrium in that subgame and this equilibrium designates actions x_i^* for each i on the equilibrium path of the play.*

Proof of Lemma 1. Let π be a game with strongly strictly Pareto efficient strategies. Let the Pareto-dominant action profile in the component game be (x_1^*, x_2^*). Take an arbitrary $\epsilon > 0$ such that for all i and all $x \in X_1 \times X_2$ with $x \neq x^*$, $\pi_i(x_i^*) > \pi_i(x) + \epsilon$.

The lower bound of the payoff from taking action x_i^* at time $\delta_i > 0$ given the opponent's current action x_{-i}^* is

$$e^{-n\lambda\delta_i}\pi_i(x^*) + (1 - e^{-n\lambda\delta_i})\underline{\pi}_i,$$

where $\underline{\pi}_i = \inf \pi_i$. The upper bound of the payoff from taking action $\hat{x}_i \neq x_i^*$ at time $\delta_i > 0$ given the opponent's current action x_{-i}^* is

$$e^{-n\lambda\delta_i}\pi_i(\hat{x}_i, x_{-i}^*) + (1 - e^{-n\lambda\delta_i})\pi(x^*).$$

Hence taking x_i^* is strictly better at time δ_i conditional on any history if

$$e^{-n\lambda\delta_i}\pi_i(x^*) + (1 - e^{-n\lambda\delta_i})\underline{\pi}_i > e^{-n\lambda\delta_i}\pi_i(\hat{x}_i, x_{-i}^*) + (1 - e^{-n\lambda\delta_i})\pi(x^*)$$

$$\begin{aligned}
&\iff e^{-n\lambda\delta_i}\left[\pi_i(x^*) - \pi_i(\hat{x}_i, x_{-i}^*)\right] > (1 - e^{-n\lambda\delta_i})\left[\pi(x^*) - \underline{\pi}_i\right]. \\
&\Longleftarrow \epsilon > \left(\frac{1}{e^{-n\lambda\delta_i}} - 1\right)\left[\pi(x^*) - \underline{\pi}_i\right]. \\
&\iff \delta_i < \frac{1}{n\lambda}\ln\left(\frac{\epsilon}{\pi(x^*) - \underline{\pi}_i} + 1\right).
\end{aligned}$$

Notice that the right hand side is strictly positive.

Let

$$\delta^* = \min_{i \in I}\left\{\frac{1}{n\lambda}\ln\left(\frac{\epsilon}{\pi(x^*) - \underline{\pi}_i} + 1\right)\right\},$$

where $I = \{1, 2\}$. Notice that δ^* is strictly positive. Also notice that for all $t \in [0, \delta^*)$, each player i chooses x_i^* at t conditional on the opponent's current action x_{-i}^*.

Now we make a backward induction argument. Suppose that for all time $t \in [0, k\delta^*)$, each player i chooses x_i^* at t conditional on the opponent's current action x_{-i}^*. We show that for all time $t' \in [k\delta^*, (k+1)\delta^*)$, each player i chooses x_i^* at t' conditional on the opponent's current action x_{-i}^*.

The lower bound of the payoff from taking action x_i^* at time $k\delta^* + \delta_i'$ with $\delta_i' > 0$ is

$$e^{-n\lambda\delta_i'}\pi_i(x^*) + (1 - e^{-n\lambda\delta_i'})\underline{\pi}_i.$$

The upper bound of the payoff from taking action $\hat{x}_i \neq x_i^*$ at time $\delta_i' > 0$ is

$$e^{-n\lambda\delta_i'}\pi_i(\hat{x}_i, x_{-i}^*) + (1 - e^{-n\lambda\delta_i'})\pi(x^*).$$

Hence taking x_i^* is strictly better at time δ_i' conditional on the opponent's current action x_{-i}^* if

$$e^{-n\lambda\delta_i'}\pi_i(x^*) + (1 - e^{-n\lambda\delta_i'})\underline{\pi}_i > e^{-n\lambda\delta_i'}\pi_i(\hat{x}_i, x_{-i}^*) + (1 - e^{-n\lambda\delta_i'})\pi(x^*)$$
$$\Longleftarrow \delta_i' < \frac{1}{n\lambda}\ln\left(\frac{\epsilon}{\pi(x^*) - \underline{\pi}_i} + 1\right).$$

Thus each player i strictly prefers playing x_i^* at all time $t' \in [k\delta^*, k\delta^* + \min_{i\in I}\left\{\frac{1}{\lambda}\ln\left(\frac{\epsilon}{\overline{\pi}_i - \underline{\pi}_i} + 1\right)\right\}) = [k\delta^*, (k+1)\delta^*)$ conditional on the opponent's current action x_{-i}^*. Since we have already proven that each player i plays x_i^* at all time $t \in [0, \delta^*)$ conditional on the opponent's current action x_{-i}^*, the backward induction argument is complete. This completes the proof. $\square$

Now we prove the proposition. Take any $\xi > 0$. At the state x where player $-i$'s move can induce x^*, player i's possible deviation is to wait forever. Also, at state x where $x_j \neq x_j^*$ for each j, player i's possible deviation is to take x_i^* and stick to it forever. Then, player $-i$ moves to x^* whenever she can move and stays at x^* if player $-i$ takes an equilibrium strategy by Lemma 1, which gives him more than $\pi_i(x^*) - \xi$ if t is sufficiently large. Therefore, on the equilibrium, the value at x is more than $\pi_i(x^*) - \xi$. Then, by feasibility, since $\pi(x^*)$ is strongly strictly Pareto efficient, player $-i$'s value at x is also close to $\pi_{-i}(x^*)$, which proves the proposition. $\square$

A.2 Proof of Proposition 2

Firstly, we prove the existence of an equilibrium with expected payoff $(1,1)$ by construction: let t^* be the solution for

$$1 = \frac{1-\exp(-2\lambda t^*)}{2}(a+1). \tag{1}$$

Note that at t^*, staying at (U,R) gives 1 to both players the payoff of 1, given that for $t \in [t^*, 0]$, $\sigma_i^t(x) = BR_i(x)$. We verify that the following strategy profile constitutes an equilibrium:

- Player 1 takes U at $-T$.
- Player 2 takes R at $-T$.
- Player 1 takes the following Markov strategy:
  - For $t \in (t^*, T)$, $\sigma_1^t(U,R) = \sigma_1^t(D,R) = U$ and $\sigma_1^t(U,L) = \sigma_1^t(D,L) = U$.
  - For $t \in [t^*, 0]$, $\sigma_1^t(x) = BR_1(x)$.
- Player 2 takes the following Markov strategy:
  - For $t \in (t^*, T)$, $\sigma_2^t(U,L) = \sigma_2^t(U,R) = R$ and $\sigma_2^t(D,L) = \sigma_2^t(D,R) = R$.
  - For $t \in [t^*, 0]$, $\sigma_2^t(x) = BR_2(x)$.

Note that at (U,R) and (D,R) at $-t$ with $t \geq t^*$, (1) ensures that the expected payoff for player 1 is 1. Similarly, at (U,L) and (U,R) at $-t$ with $t \geq t^*$, the expected payoff for player 2 is 1. Therefore,

- At $-T$, given player 2 takes R, both U and D are optimal.
- At $-T$, given player 1 takes U, both L and R are indifferent.
- For $t \in (t^*, T)$, at (U, R) and (D, R), both U and D are optimal. At (U, L) and (D, L), U is strictly optimal.
- For $t \in (t^*, T)$, at (U, L) and (U, R), both L and R are optimal. At (D, L) and (D, R), R is strictly optimal.
- For $t \in [t^*, 0]$, $\sigma_i^t(x) = BR_i(x)$ is optimal.

Secondly, we prove the existence of an equilibrium with expected payoff close to $(a, 1)$ by construction. Let $\bar{T} > 0$ be a large number. We verify that the following strategy profile constitutes an equilibrium:

- Player 1 takes U at $-T$.
- Player 2 takes R at $-T$.
- Player 1 takes the following Markov strategy:
  - for $t \in (t^*, T)$, $\sigma_1^t(U, R) = \sigma_1^t(D, R) = U$ and $\sigma_1^t(U, L) = \sigma_1^t(D, L) = U$.
  - for $t \in [t^*, 0]$, $\sigma_1^t(x) = BR_1(x)$.
- Player 2 takes the following Markov strategy:
  - for $t \in \left(\bar{T}, T\right)$, $\sigma_2^t(U, L) = \sigma_2^t(U, R) = R$ and $\sigma_2^t(D, L) = \sigma_2^t(D, R) = R$.
  - for $t \in [\bar{T}, 0]$, $\sigma_2^t(x) = BR_2(x)$.

Note that for sufficiently large $\bar{T}$, the expected payoff is sufficiently close to $(a, 1)$. We verify that the above strategy profile constitutes an equilibrium: Note that at (U, R) and (D, R) at $-t$ with $t \geq t^*$, $\bar{T} > t^*$ ensures that the expected payoff for player 1 is higher at (U,R). At (U,L) and (U,R) at $-t$ with $t \geq t^*$, the expected payoff for player 2 is 1. Therefore,

- At $-T$, given player 2 takes R, both U and D are optimal.
- At $-T$, given player 1 takes U, both L and R are indifferent.
- For $t \in (t^*, T)$, at all x, U is strictly optimal.
- For $t \in (t^*, T)$, at (U,L) and (U,R), both L and R are optimal. At (D,L) and (D,R), R is strictly optimal.
- For $t \in [t^*, 0]$, each player i takes $\sigma_i^t(x) = BR_i(x)$.

Symmetrically, the following equilibrium approximates $(1,a)$.

- Player 1 takes U at $-T$.
- Player 2 takes R at $-T$.
- Player 1 takes the following Markov strategy:
  - For $t \in \left(\bar{T}, T\right)$, $\sigma_1^t(U,R) = \sigma_1^t(D,R) = U$ and $\sigma_1^t(U,L) = \sigma_1^t(D,L) = U$.
  - For $t \in [\bar{T}, 0]$, $\sigma_1^t(x) = BR_1(x)$.
- Player 2 takes the following Markov strategy:
  - For $t \in (t^*, T)$, $\sigma_2^t(U,L) = \sigma_1^t(U,R) = R$ and $\sigma_1^t(D,L) = \sigma_1^t(D,R) = R$.
  - For $t \in [t^*, 0]$, $\sigma_2^t(x) = BR_2(x)$.

Therefore, we construct equilibria approximating $(1,1)$, $(a,1)$, and $(1,a)$. Note that the three equilibrium, nobody moves until $-\bar{T}$ and the following is also an equilibrium for any N, M: with $T^* > \bar{T}$,

- Player 1 takes U at $-T$.
- Player 2 takes R at $-T$.
- Player 1 takes the following Markov strategy:
  - For $t \in (T^*, T)$, $\sigma_1^t(U,R) = \sigma_1^t(D,R) = U$ and $\sigma_1^t(U,L) = \sigma_1^t(D,L) = U$.
  - If the number of chances where player 1 can move for $[-T, -T^*]$ is less than N, going to the first equilibrium, that is,
    - For $t \in (t^*, T^*)$, $\sigma_1^t(U,R) = \sigma_1^t(D,R) = U$ and $\sigma_1^t(U,L) = \sigma_1^t(D,L) = U$.
    - For $t \in [t^*, 0]$, $\sigma_1^t(x) = BR_1(x)$.
  - If the number of chances where player 1 can move for $[-T, -T^*]$ is no less than N and no more than M, going to the second equilibrium, that is,
    - For $t \in (t^*, T^*)$, $\sigma_1^t(U,R) = \sigma_1^t(D,R) = U$ and $\sigma_1^t(U,L) = \sigma_1^t(D,L) = U$.
    - For $t \in [t^*, 0]$, $\sigma_1^t(x) = BR_1(x)$.
  - If the number of chances where player 1 can move for $[-T, -T^*]$ is no mroe than M, going to the third equilibrium, that is,
    - For $t \in \left(\bar{T}, T^*\right)$, $\sigma_1^t(U,R) = \sigma_1^t(D,R) = U$ and $\sigma_1^t(U,L) = \sigma_1^t(D,L) = U$.
    - For $t \in [\bar{T}, 0]$, $\sigma_1^t(x) = BR_1(x)$.
- Player 2 takes the following Markov strategy:
  - For $t \in (T^*, T)$, $\sigma_2^t(U,L) = \sigma_2^t(U,R) = R$ and $\sigma_2^t(D,L) = \sigma_2^t(D,R) = R$.
  - If the number of chances where player 1 can move for $[-T, -T^*]$ is less than N, going to the first equilibrium, that is,
    - For $t \in (t^*, T^*)$, $\sigma_2^t(U,L) = \sigma_2^t(U,R) = R$ and $\sigma_2^t(D,L) = \sigma_2^t(D,R) = R$.
    - For $t \in [t^*, 0]$, $\sigma_2^t(x) = BR_2(x)$.

- If the number of chances where player 1 can move for $[-T, -T^*]$ is no less than N and no more than M, going to the second equilibrium, that is,
  - For $t \in \left(\bar{T}, T^*\right)$, $\sigma_2^t(U, L) = \sigma_2^t(U, R) = R$ and $\sigma_2^t(D, L) = \sigma_2^t(D, R) = R$.
  - For $t \in [\bar{T}, 0]$, $\sigma_2^t(x) = BR_2(x)$
- If the number of chances where player 1 can move for $[-T, -T^*]$ is no mroe than M, going to the third equilibrium, that is,
  - For $t \in (t^*, T^*)$, $\sigma_2^t(U, L) = \sigma_2^t(U, R) = R$ and $\sigma_2^t(D, L) = \sigma_2^t(D, R) = R$.
  - for $t \in [t^*, 0]$, $\sigma_2^t(x) = BR_2(x)$.

It is straightforward to show that this is an equilibrium and with appropriate choices of T^*, N, M, and sufficiently large $\left|T^* - \bar{T}\right|$, we can attain any payoff profile that can be expressed by a convex combination of $(1, 1)$, $(a, 1)$, and $(1, a)$.

A.3 Proof of Proposition 3

Proof. We use backward induction to derive the (essentially) unique subgame perfect equilibrium. Let x_i^t be the prepared action by player i at t. Firstly, note that there exists $\Delta > 0$ such that, for each $t \in [0, \Delta)$, given player $-i$'s prepared action x_{-i}^t, player i prepares the best response to x_{-i}^t, that is,

$$\sigma_t(x_{-i}^t) = BR(x_{-i}^t).$$

Suppose the players "know" the game proceeds as explained above after Δ. Then, at Δ,

- If $x_2^\Delta = L$, player 1 will take U if she can move.
- If $x_2^\Delta = R$, player 1 will take B if and only if

$$\begin{aligned}\text{The payoff of taking } B &= 1 \\ &\geq \frac{1 - \exp(-2\lambda\Delta)}{2}(a + \varepsilon + 1) \\ &= \text{The payoff of taking } U.\end{aligned}$$

Let t^* be the solution for this problem: $1 - \exp(-2\lambda t^*) = \frac{2}{a+1+\varepsilon}$.

- If $x_1^\Delta = U$, player 2 witll take L if and only if

$$\begin{aligned}\text{The payoff of taking } L &= 1 \\ &\geq \frac{1-\exp(-2\lambda\Delta)}{2}(a+1) \\ &= \text{The payoff of taking } R.\end{aligned}$$

- If $x_1^\Delta = B$, player 2 will take R.

Therefore, since $\frac{1-\exp(-2\lambda t^*)}{2}(a+1) < 1$, we can see that there exist t^* and T with $1 - \exp(-2\lambda t^*) = \frac{2}{a+1+\varepsilon}$ such that

- From $t \in (t^*, T)$,

$$\sigma_t(x_1^t) = BR(x_1^t)$$

  and

$$\begin{aligned}\sigma_t(L) &= U, \\ \sigma_t(R) &= B,\end{aligned}$$

- From $t \in (0, t^*)$, for each i,

$$\sigma_t(x_{-i}^t) = BR(x_{-i}^t).$$

Suppose the players "know" the game proceeds as follows after T. Then,

- If $x_2^T = L$, player 1 will take U if she can move.

- If $x_2^T = R$, player 1 will take B since $\frac{1-\exp(-2\lambda t)}{2}$ is increasing in t.

- If $x_1 = U$, player 2 will take L since

$$
\begin{aligned}
\text{The payoff of taking } L &= 1 \\
&> \frac{1-\exp(-2\lambda t^*)}{2}(a+1) \\
&= \text{The payoff of taking } R.
\end{aligned}
$$

- If $x_1 = D$, player 2 will take R if and only if

$$
\begin{aligned}
& \text{The payoff of taking } R \\
: \quad & = V(T) \\
: \quad & = \int_0^{T-t^*} \underbrace{\lambda \exp(-\lambda s)}_{\text{player 1 firstly moves by } t^*} \left\{ \underbrace{1-\exp\left(-\lambda\left(T-t^*-s\right)\right)}_{\text{player 2 secondly moves by } t^*} + \underbrace{\exp\left(-\lambda\left(T-t^*-s\right)\right)}_{\text{player 2 does not move by } t^*} \frac{1-\exp\left(-2\lambda t^*\right)}{2}(a+1) \right\} ds \\
& + \underbrace{\exp\left(-\lambda\left(T-t^*\right)\right)}_{\text{player 1 does not move by } t^*} a \\
\geq \quad & \underbrace{\frac{1-\exp(-2\lambda T)}{2}}_{\text{player 1 firstly moves}} + \int_0^{T-t^*} \underbrace{\exp(-\lambda s)}_{\text{player 1 does not move by } s} \underbrace{\lambda \exp(-\lambda s)}_{\text{player 2 moves at } s} V(T-s) ds + \underbrace{\exp(-2\lambda(T-t^*))}_{\text{nobody moves by } t^*} \frac{1-\exp(-2\lambda t^*)}{2} a \\
= \quad & \text{The payoff of taking } L
\end{aligned}
$$

Let T^* be the solution for the above indifference condition.

Note that

$$
\begin{aligned}
V(T) &= \int_0^{T-t^*} \lambda \exp(-\lambda s) \left\{ 1-\exp\left(-\lambda\left(T-t^*-s\right)\right) + \exp\left(-\lambda\left(T-t^*-s\right)\right) \frac{1-\exp\left(-2\lambda t^*\right)}{2}(a+1) \right\} ds \\
&\quad + \exp\left(-\lambda\left(T-t^*\right)\right) a \\
&= \int_0^{T-t^*} \lambda \exp(-\lambda s) \left\{ 1-\exp\left(-\lambda\left(T-t^*-s\right)\right) \frac{\varepsilon}{a+1+\varepsilon} \right\} ds + \exp\left(-\lambda\left(T-t^*\right)\right) a \\
&= 1 - \frac{\varepsilon}{a+1+\varepsilon} \lambda\left(T-t^*\right) \exp\left(-\left(T-t^*\right)\right) + \exp\left(-\lambda\left(T-t^*\right)\right)(a-1)
\end{aligned}
$$

$$
\begin{aligned}
& \lambda \exp\left(-2\lambda s\right) V\left(T-s\right) \\
& = \lambda \exp\left(-2\lambda s\right) - \frac{\varepsilon}{a+1+\varepsilon} \lambda^2 \left(T-s-t^*\right) \exp\left(-\lambda\left(T+s-t^*\right)\right) + (a-1)\lambda \exp\left(-\lambda\left(T+s-t^*\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&[\lambda(T-s-t^*)\exp(-\lambda(T+s-t^*))]' \\
&= -\lambda\exp(-\lambda(T+s-t^*)) - \lambda^2(T-s-t^*)\exp(-\lambda(T+s-t^*))
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^{T-t^*} -\frac{\varepsilon}{a+1+\varepsilon}\lambda^2(T-s-t^*)\exp(-\lambda(T+s-t^*))ds \\
= \ & \frac{\varepsilon}{a+1+\varepsilon}\left\{\int_0^{T-t^*}\lambda\exp(-\lambda(T+s-t^*))\,ds + [\lambda(T-s-t^*)\exp(-\lambda(T+s-t^*))]_0^{T-t^*}\right\} \\
= \ & \frac{\varepsilon}{a+1+\varepsilon}\{\exp(-\lambda(T-t^*)) - \exp(-2\lambda(T-t^*)) - \lambda(T-t^*)\exp(-\lambda(T-t^*))\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \text{The payoff of taking } L \\
= \ & \frac{1-\exp(-2\lambda T)}{2} + \int_0^{T-t^*}\lambda\exp(-2\lambda s)V(T-s)ds + \exp(-2\lambda(T-t^*))\frac{1-\exp(-2\lambda t^*)}{2}a \\
= \ & \frac{1-\exp(-2\lambda T)}{2} + \frac{1-\exp(-2\lambda(T-t^*))}{2} \\
& + \frac{\varepsilon}{a+1+\varepsilon}\{\exp(-\lambda(T-t^*)) - \exp(-2\lambda(T-t^*)) - \lambda(T-t^*)\exp(-\lambda(T-t^*))\} \\
& + (a-1)[\exp(-\lambda(T-t^*)) - \exp(-2\lambda(T-t^*))] + \exp(-2\lambda(T-t^*))\frac{1-\exp(-2\lambda t^*)}{2}a
\end{aligned}
$$

Subtracting $V(T)$ yields

$$
\begin{aligned}
& \frac{1-\exp(-2\lambda T)}{2} + \frac{1-\exp(-2\lambda(T-t^*))}{2} \\
& + \frac{\varepsilon}{a+1+\varepsilon}\{\exp(-\lambda(T-t^*)) - \exp(-2\lambda(T-t^*)) - \lambda(T-t^*)\exp(-\lambda(T-t^*))\} \\
& + (a-1)\{\exp(-\lambda(T-t^*)) - \exp(-2\lambda(T-t^*))\} + \exp(-2\lambda(T-t^*))\frac{1-\exp(-2\lambda t^*)}{2}a \\
& -1 + \frac{\varepsilon}{a+1+\varepsilon}\lambda(T-t^*)\exp(-(T-t^*)) - \exp(-\lambda(T-t^*))(a-1) \\
= \ & \exp(-\lambda T)\{\frac{-(a+1)\exp(-\lambda T))}{2} - \frac{(a-1)\exp(-\lambda T + 2\lambda t^*))}{2} \\
& + \frac{\varepsilon}{a+1+\varepsilon}\exp(\lambda t^*) - \frac{\varepsilon}{a+1+\varepsilon}\exp(-\lambda T + 2\lambda t^*))\}
\end{aligned}
$$

Note that the third term dominates when T is large.

Therefore, we have the following: suppose the players "know" the game proceeds as follows after T^*. Then, for $T > T^*$,

- If $x_2^T = L$, player 1 will take U if she can move.
- If $x_1^T = U$, player 2 witll take L since

$$\begin{aligned}
\text{The payoff of taking } L &= 1 \\
&> 1 - \exp(-\lambda(T - t^*)) + \exp(-\lambda(T - t^*))\frac{1 - \exp(-2\lambda t^*)}{2}(a + 1) \\
&= \text{The payoff of taking } R.
\end{aligned}$$

- If $x_1^T = D$, player 2 will take L
- If $x_2^T = R$, player 2 will take U.

The last line can be shown as follows. On the one hand,

$$\begin{aligned}
\text{The payoff of taking } U &= U(T) \\
&: = (1 - \exp(-\lambda(T - t^*)))(a + \varepsilon) \\
&\quad + \exp(-\lambda(T - t^*))\frac{1 - \exp(-2\lambda t^*)}{2}(a + 1 + \varepsilon) \\
&= (1 - \exp(-\lambda(T - t^*)))(a + \varepsilon) + \exp(-\lambda(T - t^*))1 \\
&= (a + \varepsilon) - \exp(-\lambda(T - t^*))(a - 1 + \varepsilon) \\
&> 1.
\end{aligned}$$

Note that $U'(T) > 0$. On the other hand, the payoff of taking D:

$$
\begin{aligned}
&\int_0^{T-T^*} \underbrace{\exp(-2\lambda s)\lambda}_{\substack{\text{1 moves first} \\ \text{going to } (U,R)}} U(T-s)ds \\
&+ \int_0^{T-T^*} \underbrace{\exp(-2\lambda s)\lambda}_{\substack{\text{2 moves first by } T^* \\ \text{going to } (D,L)}} \\
&\left[\begin{array}{c}
\underbrace{\left(1-\exp(-\lambda(T-T^*-s))\right)}_{\substack{\text{1 moves by } T^* \\ \text{going to } (U,L)}}(a+\varepsilon) \\
+\underbrace{\exp(-\lambda(T-T^*-s))}_{\text{1 does not move by } T^*} \left\{ \begin{array}{c}
\underbrace{\frac{1-\exp(-2\lambda T^*)}{2}(a+\varepsilon)}_{\substack{\text{1 moves first} \\ \text{going to } (U,L)}} + \int_0^{T^*-t^*} \underbrace{\exp(-2\lambda t)\lambda}_{\substack{\text{2 moves first by } t^* \\ \text{going to } (D,R)}} \left(\begin{array}{c}
\underbrace{\exp(-\lambda(T^*-t^*-t))}_{\substack{\text{nobody moves} \\ \text{staying at } (D,R)}} \\
+\int_0^{T^*-t^*-t} \underbrace{e^{-\lambda\tau}\lambda}_{\substack{\text{1 moves by } t^* \\ \text{going to } (D,R)}} U(T^*-t-\tau)d\tau
\end{array} \right) dt \\
+ \underbrace{\exp(-2\lambda(T^*-t^*))}_{\text{nobody moves from } (D,L) \text{ until } t^*} (1-\exp(-\lambda t^*))
\end{array} \right\}
\end{array} \right] ds \\
&+ \underbrace{\exp(-2\lambda\left(T-T^*\right))}_{\substack{\text{nobody moves until } T^* \\ \text{going to } (D,R)}} \left\{ \begin{array}{c}
\underbrace{\exp(-\lambda(T^*-t^*))}_{\substack{\text{nobody moves} \\ \text{staying at } (D,R)}} \\
+\int_0^{T^*-t^*} \underbrace{\exp(-\lambda\tau)\lambda}_{\substack{\text{1 moves by } t^* \\ \text{going to } (D,R)}} V(T^*-\tau)d\tau
\end{array} \right\}
\end{aligned}
$$

- The first term is

$$
\begin{aligned}
& \int_0^{T-T^*} \exp(-2\lambda s)\lambda U(T-s)ds \\
= & \int_0^{T-T^*} \exp(-2\lambda s)\lambda \left((a+\varepsilon) - \exp(-\lambda(T-s-t^*))(a-1+\varepsilon)\right) ds \\
= & (a+\varepsilon)\frac{1-\exp(-2\lambda\left(T-T^*\right))}{2} \\
& - (a-1+\varepsilon)\left(\exp\left(-\lambda\left(T-t^*\right)\right) - \exp(-\lambda\left(T-t^*+T-T^*\right))\right)
\end{aligned}
$$

- The second term is

$$
\begin{aligned}
&\int_0^{T-T^*} \exp(-2\lambda s)\lambda \\
&\left[\begin{array}{c}
\left(1-\exp(-\lambda(T-T^*-s)\right)(a+\varepsilon) \\
+\exp(-\lambda(T-T^*-s)\left\{\begin{array}{c}
\frac{1-\exp(-2\lambda T^*)}{2}(a+\varepsilon)+\exp\left(-\lambda\left(T^*-t^*\right)\right)-\exp\left(-2\lambda\left(T^*-t^*\right)\right) \\
+(a+\varepsilon)\left(\frac{1-\exp(-2\lambda(T^*-t^*))}{2}-\exp\left(-\lambda\left(T^*-t^*\right)\right)+\exp\left(-2\lambda\left(T^*-t^*\right)\right)\right) \\
-\left(T^*-t^*\right)\lambda\exp\left(-\lambda\left(T^*-t^*\right)\right)(a-1+\varepsilon)+\exp\left(-\lambda\left(T^*-t^*\right)\right)\left(1-\exp\left(-\lambda\left(T^*-t^*\right)\right)\right)(a-1+\varepsilon) \\
+\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*))
\end{array}\right\}
\end{array}\right] ds \\
= &\int_0^{T-T^*}\left[\begin{array}{c}
\left(\exp(-2\lambda s)\lambda-\exp(-\lambda(T-T^*+s))\lambda\right)(a+\varepsilon) \\
+\exp(-\lambda(T-T^*+s)\left\{\begin{array}{c}
\frac{1-\exp(-2\lambda T^*)}{2}(a+\varepsilon)+\exp\left(-\lambda\left(T^*-t^*\right)\right)-\exp\left(-2\lambda\left(T^*-t^*\right)\right) \\
+(a+\varepsilon)\left(\frac{1-\exp(-2\lambda(T^*-t^*))}{2}-\exp\left(-\lambda\left(T^*-t^*\right)\right)+\exp\left(-2\lambda\left(T^*-t^*\right)\right)\right) \\
-\left(T^*-t^*\right)\lambda\exp\left(-\lambda\left(T^*-t^*\right)\right)(a-1+\varepsilon) \\
+\exp\left(-\lambda\left(T^*-t^*\right)\right)\left(1-\exp\left(-\lambda\left(T^*-t^*\right)\right)\right)(a-1+\varepsilon) \\
+\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*))
\end{array}\right\}
\end{array}\right] ds \\
= &\frac{1-\exp\left(-2\lambda\left(T-T^*\right)\right)}{2}(a+\varepsilon) \\
&+\exp(-\lambda(T-T^*)\left(1-\exp(-\lambda(T-T^*)\right)\left(\begin{array}{c}
-(a+\varepsilon)+\frac{1-\exp(-2\lambda T^*)}{2}(a+\varepsilon)+\exp\left(-\lambda\left(T^*-t^*\right)\right)-\exp\left(-2\lambda\left(T^*-t^*\right)\right) \\
+(a+\varepsilon)\left(\frac{1-\exp(-2\lambda(T^*-t^*))}{2}-\exp\left(-\lambda\left(T^*-t^*\right)\right)+\exp\left(-2\lambda\left(T^*-t^*\right)\right)\right) \\
-\left(T^*-t^*\right)\lambda\exp\left(-\lambda\left(T^*-t^*\right)\right)(a-1+\varepsilon) \\
+\exp\left(-\lambda\left(T^*-t^*\right)\right)\left(1-\exp\left(-\lambda\left(T^*-t^*\right)\right)\right)(a-1+\varepsilon) \\
+\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*))
\end{array}\right) \\
= &\frac{1-\exp\left(-2\lambda\left(T-T^*\right)\right)}{2}(a+\varepsilon) \\
&+\exp(-\lambda(T-T^*)\left(1-\exp(-\lambda(T-T^*)\right)\left(\begin{array}{c}
-(a+\varepsilon) \\
+(a+\varepsilon)\left\{1-\exp(-2\lambda\left(T-t^*\right))\right\}-(a-1+\varepsilon)\left\{\exp\left(-\lambda\left(T^*-t^*\right)\right)+\exp\left(-2\lambda\left(T^*-t^*\right)\right)\right\} \\
-\left(T^*-t^*\right)\lambda\exp\left(-\lambda\left(T^*-t^*\right)\right)(a-1+\varepsilon) \\
+\exp\left(-\lambda\left(T^*-t^*\right)\right)\left(1-\exp\left(-\lambda\left(T^*-t^*\right)\right)\right)(a-1+\varepsilon) \\
+\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*))
\end{array}\right)
\end{aligned}
$$

since

$$
\begin{aligned}
& \int_0^{T^*-t^*} \exp(-2\lambda t)\lambda \left(\exp(-\lambda(T^*-t^*-t)) + \int_0^{T^*-t^*-t} \exp(-\lambda\tau)\lambda U(T^*-t-\tau)d\tau \right) dt \\
= & \int_0^{T^*-t^*} \exp(-2\lambda t)\lambda \exp(-\lambda(T^*-t^*-t))dt + \exp(-2\lambda t)\lambda \cdot \\
& \left(\int_0^{T^*-t^*-t} \exp(-\lambda\tau)\lambda \left\{ (a+\varepsilon) - \exp(-\lambda(T^*-t-\tau-t^*))(a-1+\varepsilon) \right\} d\tau \right) dt \\
= & \int_0^{T^*-t^*} \lambda \exp(-\lambda(T^*-t^*+t))dt + \exp(-2\lambda t)\lambda[\left\{1-\exp(-\lambda\left(T^*-t^*-t\right))\right\}(a+\varepsilon) \\
& -\left(T^*-t^*-t\right)\exp(-\lambda(T^*-t-t^*))(a-1+\varepsilon)]dt \\
= & \exp\left(-\lambda\left(T^*-t^*\right)\right) - \exp\left(-2\lambda\left(T^*-t^*\right)\right) \\
& + (a+\varepsilon)\left(\frac{1-\exp(-2\lambda\left(T^*-t^*\right))}{2} - \exp\left(-\lambda\left(T^*-t^*\right)\right) + \exp\left(-2\lambda\left(T^*-t^*\right)\right) \right) \\
& - \left(T^*-t^*\right)\lambda \exp\left(-\lambda\left(T^*-t^*\right)\right)(a-1+\varepsilon) \\
& + \exp\left(-\lambda\left(T^*-t^*\right)\right)\left(1-\exp\left(-\lambda\left(T^*-t^*\right)\right)\right)(a-1+\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^{T^*-t^*-t} \exp(-\lambda\tau)\lambda U(T^*-t-\tau)d\tau \\
= & \int_0^{T^*-t^*-t} (a+\varepsilon)\exp\left(-\lambda\tau\right)\lambda - \exp\left(-\lambda\left(T^*-t^*-t\right)\right)\lambda\left(a-1+\varepsilon\right)d\tau \\
= & (a+\varepsilon)\left(1-\exp\left(-\lambda\left(T^*-t^*-t\right)\right)\right) - \left(T^*-t^*-t\right)\exp\left(-\lambda\left(T^*-t^*-t\right)\right)\lambda\left(a-1+\varepsilon\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left[\left(T^*-t^*-t\right)\lambda \exp\left(-\lambda\left(T^*-t^*+t\right)\right)(a-1+\varepsilon)\right]' \\
& = -\lambda \exp\left(-\lambda\left(T-t^*+t\right)\right)(a-1+\varepsilon) - \lambda^2\left(T^*-t^*-t\right)\exp\left(-\lambda\left(T^*-t^*+t\right)\right)(a-1+\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
& \left[\left(T^*-t^*-t\right)\lambda \exp\left(-\lambda\left(T^*-t^*+t\right)\right)(a-1+\varepsilon)\right] + \int \lambda \exp\left(-\lambda\left(T^*-t^*+t\right)\right)(a-1+\varepsilon) \\
& = -\int_0^{T^*-t^*} \lambda^2\left(T^*-t^*-t\right)\exp\left(-\lambda\left(T^*-t^*+t\right)\right)(a-1+\varepsilon)dt
\end{aligned}
$$

$$(-(T^*-t^*)\lambda\exp(-\lambda(T^*-t^*))+(1-\exp(-\lambda(T^*-t^*))))\exp(-\lambda(T^*-t^*))(a-1+\varepsilon)$$

- The last term is

$$\begin{aligned}
& \exp(-2\lambda(T-T^*))\left\{\exp(-\lambda(T^*-t^*))+\int_0^{T^*-t^*}\exp(-\lambda\tau)\lambda U(T^*-\tau)d\tau\right\} \\
= \; & \exp(-2\lambda(T-T^*)) \\
& \left\{\exp(-\lambda(T^*-t^*))+\int_0^{T^*-t^*}\exp(-\lambda\tau)\lambda\left[(a+\varepsilon)-\exp(-\lambda(T^*-\tau-t^*))(a-1+\varepsilon)\right]d\tau\right\} \\
= \; & \exp(-2\lambda(T-T^*))\{\exp(-\lambda(T^*-t^*)) \\
& +\{1-\exp(-\lambda(T^*-t^*))\}(a+\varepsilon)-\lambda(T^*-t^*)\exp(-\lambda(T^*-t^*))(a-1+\varepsilon)\} \\
= \; & \exp(-2\lambda(T-T^*)) \\
& \{(a+\varepsilon)-\exp(-\lambda(T^*-t^*))(a-1+\varepsilon)-\lambda(T^*-t^*)\exp(-\lambda(T^*-t^*))(a-1+\varepsilon)\} \\
= \; & \exp(-2\lambda(T-T^*))(a+\varepsilon) \\
& -\exp(-2\lambda(T-T^*))(\lambda(T^*-t^*)+1)\exp(-\lambda(T^*-t^*))(a-1+\varepsilon)
\end{aligned}$$

Hence, the payoff of taking D is

$$\begin{aligned}
& (a+\varepsilon)\frac{1-\exp(-2\lambda(T-T^*))}{2}-(a-1+\varepsilon)(\exp(-\lambda(T-t^*))-\exp(-\lambda(T-t^*+T-T^*))) \\
& +\frac{1-\exp(-2\lambda(T-T^*))}{2}(a+\varepsilon)+\exp(-\lambda(T-T^*)(1-\exp(-\lambda(T-T^*))\cdot \\
& \left(\begin{array}{c}
-(a+\varepsilon) \\
+(a+\varepsilon)\left\{1-\frac{\exp(-2\lambda(T^*-t^*))}{2}-\frac{\exp(-2\lambda T^*)}{2}\right\} \\
-(a-1+\varepsilon)\{\exp(-\lambda(T^*-t^*))+\exp(-2\lambda(T^*-t^*))\} \\
-(T^*-t^*)\lambda\exp(-\lambda(T^*-t^*))(a-1+\varepsilon) \\
+\exp(-\lambda(T^*-t^*))(1-\exp(-\lambda(T^*-t^*)))(a-1+\varepsilon) \\
+\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*))
\end{array}\right) \\
& +\exp(-2\lambda(T-T^*))(a+\varepsilon)-\exp(-2\lambda(T-T^*))(\lambda(T^*-t^*)+1)\exp(-\lambda(T^*-t^*))(a-1+\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&(a+\varepsilon) - (a-1+\varepsilon)\left(\exp\left(-\lambda\left(T-t^*\right)\right) - \exp(-\lambda\left(T-t^*+T-T^*\right))\right) \\
&+\exp(-\lambda(T-T^*)\left(1-\exp(-\lambda(T-T^*)\right)\left(\begin{array}{c} -\frac{a+\varepsilon}{2}\left(\exp(-2\lambda\left(T^*-t^*\right)) + \exp(-2\lambda T^*)\right) \\ +\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*)) \end{array}\right) \\
&-\exp(-\lambda(T-T^*))\left(T^*-t^*\right)\lambda\exp\left(-\lambda\left(T^*-t^*\right)\right)\left(a-1+\varepsilon\right)
\end{aligned}
$$

Subtracting

$$
(a+\varepsilon) - \exp(-\lambda(T-t^*))(a-1+\varepsilon)
$$

yields

$$
\begin{aligned}
&+\exp(-\lambda(T-T^*)\left(1-\exp(-\lambda(T-T^*)\right)\left(\begin{array}{c} -\frac{a+\varepsilon}{2}\left(\exp(-2\lambda\left(T^*-t^*\right)) + \exp(-2\lambda T^*)\right) \\ +\exp(-2\lambda(T^*-t^*))(1-\exp(-\lambda t^*)) \end{array}\right) \\
&-\exp(-\lambda(T-T^*))\left(T^*-t^*\right)\lambda\exp\left(-\lambda\left(T^*-t^*\right)\right)\left(a-1+\varepsilon\right) < 0
\end{aligned}
$$

□

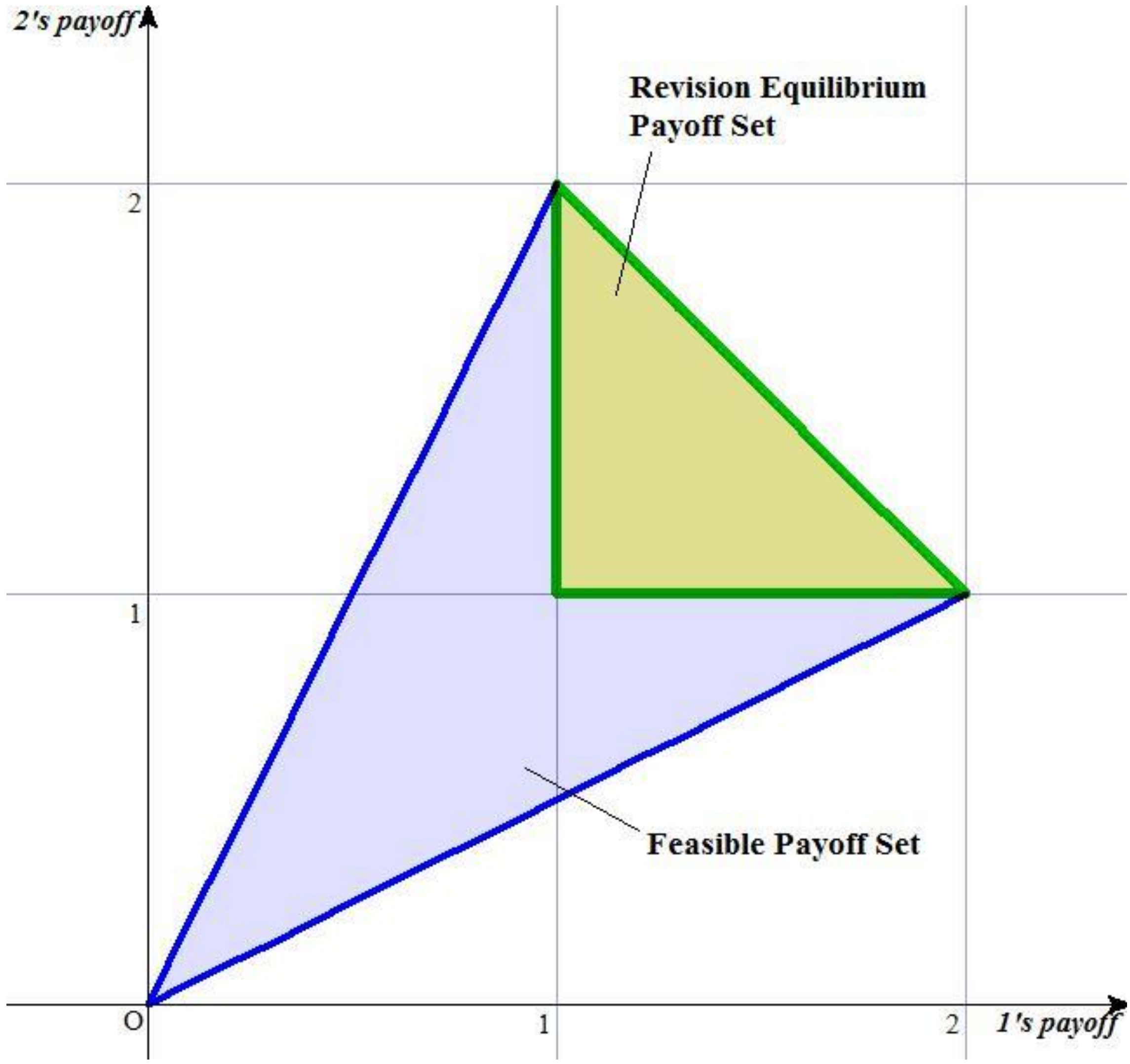

Figure 1: Feasible payoff set and revision equilibrium payoff set in symmetric "battle of the sexes" game with a=2.

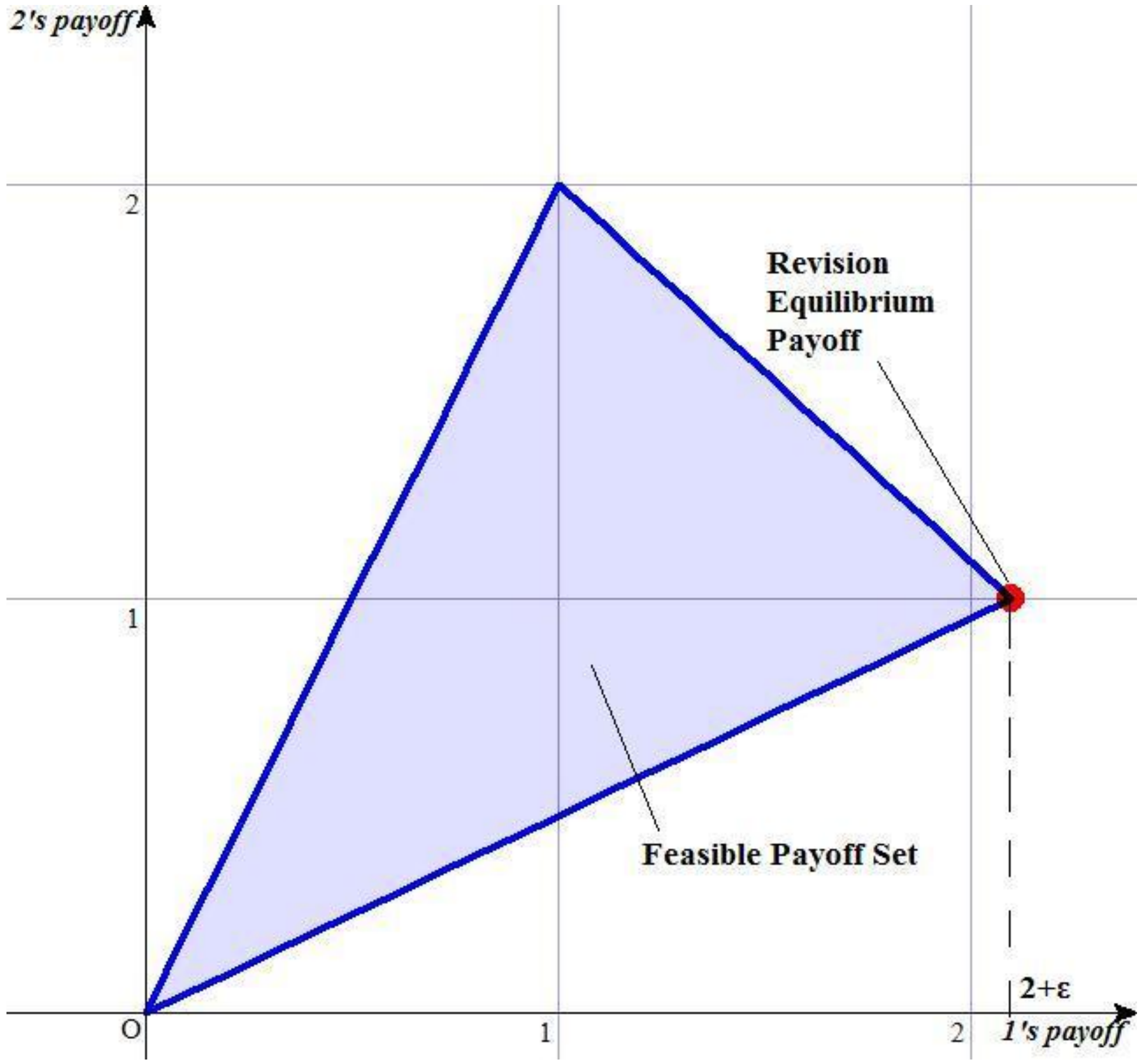

Figure 2: Feasible payoff set and revision equilibrium payoff in asymmetric "battle of the sexes" game with a=2.