Coalitional Network Games [†]

Preliminary version

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Abstract

Coalitional network games are real-valued functions defined on a set of players (the society) organized into a network and a coalition structure. The network specifies the nature of the relationship each individual has with the other individuals and the coalition structure specifies a collection of groups among the society. Coalitional network games model situations where the total productive value of a network among players depends on the players' group membership. These games thus capture the public good aspect of bilateral cooperation, i.e., network games with externalities. After studying the specific structure of coalitional networks, we propose an allocation rule under the perspective that players can alter the coalitional network structure. This means that the value of all potential alternative coalitional networks can and should influence the allocation of value among players in any given coalitional network structure.

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1 Introduction

Consider situations where individuals or players from a society negotiate bilaterally to cooperate. In such cases individuals are connected in some network relationship. Consider at the same time that the individuals involved in a network relationship belong to groups, clubs, or coalitions. Many applications are instances of such cooperative relationships, ranging from friendships among people belonging to different communities, to communicating information about job opportunities among people with different skills, to strategic alliances between firms from different groups, to bilateral free trade agreements among countries from different free trade zones. A common feature of these situations is that the way in which players are connected to each other and the way they are organized in mutually disjoint groups determine the total productivity or value generated by the society. This total productivity or value may be captured by a real-valued function defined on structures consisting of a network and a collection of disjoint coalitions. Such structures are named *coalitional networks*, and the real-valued functions defined on coalitional networks are named *coalitional network games* (see Caulier, Mauleon and Vannetelbosch, 2009).

In this article we examine how the total value generated by players cooperating through networks and groups should be allocated or transferred among players. This issue appears to be important both in terms of equity : what is a player's fair share to her cooperative participation ; and also in terms of players' incentives to cooperate : what is the player's prospect to cooperate in a particular coalitional network ? Put together these two questions can be summarized by the following one : what kind of coalitional network structure can we expect to form ?

Cooperative game theory literature provides rich axiomatic foundations on the allocation of value by examining which productive value is generated by each possible coalition of players. In coalitional network games, primitive informations are different from cooperative game theory with respect to two aspects. First, the value generated by a group of players depends not only on the identities of the players but also on the links connecting the players. Second, we cannot examine the value generated by a group of players independently of how are organized the other players. In cooperative game theory, the value generated by a group or coalition of players is independent of the coalition structure formed by the other players. In our setting, a value is generated by the whole society and depend on how the players are connected to each other and how players are organized in a coalition structure. In particular, the value generated by a network among players is influenced by the structure of coalition they form, but also by the coalition structure formed by the remaining players in the society. Coalitional network games can thus model any situation where the value generated by a network is influenced by the presence of externalities.

The first aspect differentiating coalitional network games from cooperative games is tackled in Jackson and Wolinsky (1996). In their paper, they develop the notion of network games, a richer object than cooperative games in the sense that the value achieved by a set of players depends directly on how these players are connected by a network relationship, rather than just the identities of players in the set. The setting they propose is rich enough to allow for costs and benefits to emerge differently according to different organization of links in a network, and allows for externalities across players and networks.

The second aspect is tackled in Gilboa and Lehrer (1991). In their paper, they develop the notion of global games, a richer object than cooperative games in the sense that the value achieved by a set of players depends on how players outside this set are organized in term of coalitions. There, the primitive information is which value is produced by each possible partition[1] of the society. In global games, a coalition cannot ensure the production of a value independently of how are grouped the other players. Their setting captures the public good aspect of cooperation.[2]

In our approach, we want to tackle both aspects at the same time : players may form links and a coalition structure (partition) among themselves. Our primitive information is thus the productive value that accrues to each possible network in conjunction with a partition, i.e., a coalitional network.

Moreover, we adopt the *flexible* perspective first proposed by Jackson (2005) in the context of network games : a coalitional network structure is not fixed but is something that can be altered or still to be shaped. From the perspective of the value allocation, it means that its computation should take into account all the other potential alternative coalitional network structures and not just the coalitional network that actually forms and its subcoalitional networks. In evaluating players'

[1] A partition is a collection of mutually disjoint sets whose union forms the society.

[2] There exist other approaches to cooperative games that allow for externalities, such as games in partition function form (Thrall and Lucas, 1963). But in partition function form games only embedded coalition structures are allowed (see Grabisch, 2009 for details).

contribution to value, we want to assess the contribution to value of other potential coalitional networks that could have been achieved and if some other players might have served as substitute (see Jackson, 2005). As in Jackson (2005), we adopt the flexible approach because in some cases, the efficient coalitional network won't be the complete one. Indeed, in cooperative game theory, in order to allocate a value across players, a rule such as the Shapley value decomposes the grand coalition in various ways to evaluate players' contributions. It is thus explicitly assumed that the grand coalition is the ultimate structure that forms whose value has to be allocated. Hence, in the decomposition of the grand coalition, the value of every other coalition is taken into account in the computation of players' contributions. Because in coalitional network games the complete coalitional network needs not to be the ultimate one that forms, we thus have to care about allocating the value of a coalitional network different than the complete one. In order to accomplish this task, we must gather information about coalitional networks that are not subcoalitional networks of a given coalitional network.

The paper is structured as follows. Section 2 provides definitions for coalitional networks. Section 3 presents the lattice structure of coalitional networks. Section 4 introduces coalitional network games. Section 5 proposes a new allocation rule and characterizes it.

2 Networks and partitions

Let a society be a finite set $N = \{1, \dots n\}$ of players. We consider N as fixed throughout the paper. A technology is available in the society but requires bilateral collaboration among the players in order to produce a value. Each player negotiates bilaterally with another player in order to collaborate in a production activity. These collaborations are modeled by a list of pairs of players linked to each other and are represented by an undirected graph. A link between two players $i, j \in N$, $i \neq j$, is denoted ij or ji and means that players i and j have agreed to collaborate. For notational convenience, when the identities of linked players is not needed, we sometimes use the generic symbol ℓ for a link.

A *network* on a society N is a collection of links and is denoted g. The complete network g^N is the network in which each player is connected to all the other players. Similarly, we denote by g^S the complete network among players in a subset S of

N. $G = \{g \mid g \subseteq g^N\}$ denotes the set of all possible networks on the society N. The empty network $g^\emptyset \in G$ depicts the situation where all players are isolated and means that no collaboration agreement has been reached.

Let g denote a network in the society and S a strict subset of N. Then $g_{|S} = \{ij \in g \mid i \in S,\ j \in S\}$ is the network obtained from g by deleting all links with players outside S. Let $N(g)$ denote the set of players with at least one link in the network g. The cardinality of $N(g)$ is denoted $n(g)$. Similarly, $N_i(g) = \{ij \in g \mid j \in N\}$ denotes the set of links involving i in g. The notation $n_i(g)$ has obvious meaning. We denote by $|g|$ the number of links in a network $g \in G$. We have $|g^N| = \frac{n(n-1)}{2}$.

A *path* in a network $g \in G$ between two players i and j is a sequence of players $i_1, \dots, i_K$ such that $i_k i_{k+1} \in g$ for any $k \in \{1, \dots, K-1\}$, with $i_1 = i$ and $i_K = j$. If there exists a path between players i and j, $i \neq j$, then i is said to be (indirectly) connected to j. Path connection in a network g offers a natural equivalence relation between players and thus, is a partition of g (and eventually of N). The elements of this kind of partition are the *components* of the network g. A *component* of a network g is a subnetwork $g' \subseteq g$ such that

(i) if $i \in N(g')$ and $j \in N(g')$, $i \neq j$, then there exists a path between i and j,

(ii) if $ij \in g$ and $i \in N(g')$, then $ij \in g'$.

The set of components $C(g)$ of network g is a partition of g in distinct connected subnetworks:

- $g = \bigcup_{g' \in C(g)} g'$,
- $g' \cap g'' = g^\emptyset$, $g' \in C(g)$, $g'' \in C(g)$,
- $g' \neq g^\emptyset$ for all $g' \in C(g)$.

Since an isolated player is not considered as a component under the above definition, in order to obtain a partition of N from the set of components it is enough to consider isolated players as singletons. We denote by $P^C(g) = \{S^1, \dots, S^m\}$ the partition of N such that $S^k \in P^C(g)$ with $S^k = N(g')$, $g' \in C(g)$ or $S^k = \{i\}$.

In the environment we study, cooperation also occurs under coalition formation : players may group themselves in mutually disjoint coalitions. Players may group themselves on an *ad hoc* basis, only for cooperation purpose or players may be characterized by some idiosyncratic attributes and players with similar attributes group themselves in the same coalition. For example, players in the society may be endowed with different skills or that a given skill is observed in different levels among players. A collection of groups in a society is represented by a *coalition structure* or *partition*. The elements of a partition are called *blocks* or if we refer to the partition as a coalition structure, the elements are called *groups* or *coalitions*. A *k-partition* is a partition that consists of k blocks. Formally, we denote a coalition structure $P = \{S^1, ..., S^m\}$ such that $S^k \neq \emptyset$ for $k = 1, \ldots, m$, $S^k \cap S^{k'} = \emptyset$, $k \neq k'$ and $\bigcup_k S^k = N$. The interpretation is the following: players i and j share the same characteristic if and only if they belong to the same coalition. For instance, if the characteristic of interest is path connection in a network g, then $P = P^C(g)$. We denote by $\mathscr{P}$ the set of possible coalition structures in the society.

A society in which players belong to groups and are connected in some network relationship is called a *coalitional network*. Formally, a coalitional network consists of a pair $(g, P) \in G \times \mathscr{P}$.

Before turning to the study of the allocation of value generated by coalitional networks, we first present the lattice structure of coalitional networks.

3 The lattice of coalitional networks

We begin with some very general definitions about lattices.

3.1 Lattices

A *lattice*[3] is a set L endowed with a reflexive, antisymmetric and transitive partial order binary relation $\leq$ such that for any two elements $x, y \in L$, there is a unique least upper bound (supremum), denoted by $x \vee y$ and a unique greatest lower bound (infimum) denoted $x \wedge y$. We call the lattice element $x \vee y$ the *join* of x and y and element $x \wedge y$ the *meet* of x and y. For any $S \subseteq L$, $S = \{x_1, \ldots, x_n\}$, we denote $\bigvee_{1 \leq i \leq n} x_i = x_1 \vee x_2 \vee \cdots \vee x_n$ the least upper bound of S. We define $\bigwedge_{1 \leq i \leq n} x_i$ the greatest lower bound of S similarly. The elements $\perp$ and $\top$ of a lattice L such that $\perp \leq x$ and $x \leq \top$ for all $x \in L$ are, respectively, the bottom and top element. Other binary relations can be deduced from $\leq$. The antisymmetric part of $\leq$ is defined by

[3]We restrict our attention to finite lattices.

$x < y \Leftrightarrow x \leq y, x \neq y$. For all $x, y \in L$, y *covers* x, denoted $x \prec y$, if $x \leq z \leq y$ implies $z = x$ or $y = z$. The dual of a lattice $(L, \leq)$ is the lattice $(L^*, \geq)$ such that $x \leq y$ in L iff $x \geq y$ in L^*.

A *chain* between two elements x and y is a sequence of elements $\{x = x_1, x_2, \ldots, x_n = y\} \subset L$ such that $x_1 \prec x_2 \prec \cdots \prec x_n$. The *length* of a chain is the number of elements in the chain, minus 1. A chain between two elements is *maximal* if any other chain between the two elements has shorter length. If all maximal chains between any two elements of L have the same length, then L satisfies the *Jordan-Dedekind condition* (JD-condition). The *rank* in a lattice satisfying the JD-condition is an integer-value function r with domain L such that $r(\perp) = 0$ and $r(y) = r(x) + 1$ iff $x \prec y$. Conversely, a lattice admits a rank function if the lattice satisfies the JD-condition. It is thus implicit that a lattice satisfies the JD-condition when we talk about the rank of the lattice. The rank of an element x of L can be interpreted as the length of any maximal chain between $\perp$ and x. A lattice element a such that $r(a) = 1$ is called an *atom*, i.e. $\perp \prec a$. We denote $\mathscr{A}(L)$ the set of atoms of L.

A lattice is *atomic* if every element x can be expressed as $\bigvee_{a \in S} a = x$, $S \subseteq \mathscr{A}(L)$. The expression $x = a_1 \vee \cdots \vee a_k$ with $a_i \in \mathscr{A}(L)$ for all i, is an atom *decomposition* of x. A decomposition is *irredundant* if $x \neq a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_k$ for any $i = 1, \ldots, k$. A set $S \subseteq \mathscr{A}(L)$ is an irredundant decomposition if it has *minimal size* as decomposition. A decomposition always exists in an atomic lattice (with bottom element $\perp$), and thus an irredundant decomposition always exists by deleting superfluous elements. In most lattices there exist several distinct irredundant decompositions. For any element x in an atomic lattice L, let $ID(x) = \big\{S_1, \ldots, S_m \mid \text{for all } i = 1, \ldots, m : \bigvee_{a \in S_i} a = x, S_i \text{ minimal and } \{S_i\}_{i=1}^m \in \mathscr{A}(x)\big\}$, the set of irredundant decompositions as supremum of atoms of x. For each element x in an atomic lattice, there exists a unique (may be redundant) *maximal decomposition* as supremum of atoms that involves all atoms below the element.

Formally :

$$x = \bigvee_{a \in \mathscr{A}(x)} a$$

with

$$\mathscr{A}(x) = \{a \in \mathscr{A}(L) \mid a \leq x\}$$

and thus

$$\mathscr{A}(x) = \left\{ \bigcup_{i=1}^{m} S_i \mid S_i \in ID(x) \right\}.$$

An element x in a lattice L is *join-irreducible* if x cannot be expressed as $x = \bigvee_{y \in S} y$ for any $S \subset L$, i.e. x covers only one element. A lattice whose only join-irreducible elements are precisely the set of atoms is *atomistic*. Note that atomistic lattices are atomic but the converse is not necessarily true.

A lattice L is *upper semimodular* or simply *supermodular* if for all $x, y \in L$, $x \wedge y \prec x$ and $x \wedge y \prec y$ imply $x \prec x \vee y$ and $y \prec x \vee y$. It is well known that a finite semimodular lattice satisfies the JD-condition and thus possesses a rank function (see Stern (1999), Theorem 1.9.1). For semimodular lattices, the rank function satisfies the following property :

Theorem 3.1 *A lattice L is semimodular if and only if its rank function $r : L \longrightarrow \mathbb{N}$ satisfies*

$$r(x) + r(y) \geq r(x \vee y) + r(x \wedge y) \tag{1}$$

for all $x, y \in L$.

Proof. See Stern (1999), Theorem 1.9.9. ■

3.2 Lattices on networks and lattices on partitions

The set of possible networks G has a lattice structure under the network-inclusion relation with infimum and supremum given by $g \wedge g' = g \cap g'$ and $g \vee g' = g \cup g'$, $g, g' \in G$. Bottom element is $g^{\emptyset}$ and top element is g^N. The atoms $\mathscr{A}(G)$ are the one-link networks $\ell \subset g^N$. The lattice $(G, \subseteq)$ is ranked (maximal chains have the same length) and each element $g \in G$ has rank $r(g) = |g|$, i.e. the rank of a network g is precisely the number of links in g. The rank of any network corresponds to the *degree*, i.e. its number of links and also corresponds to the number of atoms below the network. Hence, we identify the number of atoms below a network g as the degree of g. Observe that if a network g covers a network g' then there exists a network $a \in \mathscr{A}(G)$ such that $g' \vee a = g$ and $r(g) = r(g') + 1$, the network g has one more link than g'.

The set of possible coalition structures or partitions $\mathscr{P}$ has a lattice structure under the *refinement* ordering $\sqsubseteq$. Let P, P' be partitions of N. We say that P is a refinement of P', denoted $P \sqsubseteq P'$ if any block of P is a subset of a block of P'. The dual relation of the refinement is the *coarsening* relation. The infimum between two partitions P and P' is $P \wedge P'$ and is defines as the coarsest partition finer than both

P and P'. The supremum between two partitions P and P' is $P \vee P'$ and is defined as the finest partition coarser than both P and P'. The bottom element of the partition $(\mathscr{P}, \sqsubseteq)$ is the *finest* partition $P_\perp := \{\{1\}, \ldots, \{n\}\}$. The top element, i.e. the *coarsest* partition, is the grand coalition $P^\top := \{N\}$. The atoms $\mathscr{A}(\mathscr{P})$ are the partitions whose only non-trivial block is a two-element coalition : $Q_{ij} \in \mathscr{A}(\mathscr{P})$ if there exist $i, j \in N$ such that $\{i, j\} \in Q_{ij}$ and all other blocks of Q_{ij} are singletons. The lattice $(\mathscr{P}, \sqsubseteq)$ is ranked and each element $P \in \mathscr{P}$ has rank $r(P) = n - |P|$, with $|P|$ the number of blocks in $P = \{S_1, \ldots, S_{|P|}\}$. For any $P, P' \in \mathscr{P}$ such that P is covering P', we have $r(P') = r(P) + 1$, P' has one more block than P.[4]

The *class* of a partition $P \in \mathscr{P}$ is defined by the collection of integers $c^P = \{c_1^P, \ldots, c_n^P\}$ such that c_k^P is the number of blocks of P consisting of exactly k players. Thus $\sum_{k=1}^n c_k^P k = n$ and $\sum_{k=1}^n c_k^P = n - r(P) = |P|$ (see Rossi [9]).

The *size* s^P of a partition $P \in \mathscr{P}$ is the number of atoms below P. Formally :

$$s^P = \sum_{k=1}^{n} c_k^P \binom{k}{2} = |\{\{i, j\} \in \mathscr{A}(P)\}| \tag{2}$$

with $\mathscr{A}(P) = \{Q_{ij} \in \mathscr{A}(\mathscr{P}) \mid Q_{ij} \sqsubseteq P\}$.

In this context, the size of a partition P is the number of atoms below P and constitutes its maximal decomposition as supremum of atoms.

3.3 Lattices on coalitional networks

The set of possible coalitional networks on the society N is the cartesian product of lattices G and $\mathscr{P}$: $G \times \mathscr{P} := \{(g, P) \mid g \in G, P \in \mathscr{P}\}$. Define the ordering relation $\leq$ on $G \times \mathscr{P}$ such that $(g, P) \leq (g', P')$ in $G \times \mathscr{P}$ if $g \subseteq g'$ in G and $P \sqsubseteq P'$ in $\mathscr{P}$.

Proposition 3.1 *The set of possible coalitional networks $G \times \mathscr{P}$ endowed with the binary relation $\leq$ such that $(g, P) \leq (g', P')$ in $G \times \mathscr{P}$ if $g \subseteq g'$ in G and $P \sqsubseteq P'$ in $\mathscr{P}$ is a lattice.*

Proof. For each pair (g, P),(g', P') in $G \times \mathscr{P}$, there exists a unique least upper bound $(g, P) \vee (g', P') = (g \cup g', P \cup P')$ and a unique greatest lower bound $(g, P) \wedge (g', P') = (g \cap g', P \cap P')$ since, by the lattice property of G and $\mathscr{P}$, $g \cup g'$, $g \cap g'$, $P \cup P'$ and $P \cap P'$ are unique. ■

[4]Any partition P' covered by P have the same blocks as P but one, which is divided in 2 blocks in P'.

The bottom and top elements of the lattice $(G\times\mathscr{P},\leq)$ are $(g^{\emptyset},P_{\perp})$ and $(g^N,\{N\})$ respectively. Atom elements in $\mathscr{A}(G\times\mathscr{P})$ take one of the following two forms : $(\ell,P_{\perp})$ or $(g^{\emptyset},Q_{ij})$ with $\ell\in G$ a one-link network and $Q_{ij}\in\mathscr{A}(\mathscr{P})$. Direct calculations show $|\mathscr{A}(G\times\mathscr{P})|=\left[\frac{n(n-1)}{2}\right]+\binom{n}{2}=n(n-1)$.

Each element (g,P), with $P:=\{S_1,\ldots,S_k\}$ a k-partition, is covered by $\binom{k}{2}+\left(|g^N|-|g|\right)$ elements and covers $\sum_{S\in P}2^{|S|-1}-|P|+|g|$ elements.

The number of atoms in a maximal decomposition as supremum of atoms of any (g,P) is $|\mathscr{A}(g,P)|=s^P+|g|$ with s^P defined in (2). We call $|\mathscr{A}(g,P)|$ the *degree* of the coalitional network (g,P) and denote it by $d(g,P)$. For any player $i\in N$ and $(g,P)\in G\times\mathscr{P}$, we denote $d_i(g,P)$ the *degree* of player i in the coalitional network (g,P). The degree of a player in a coalitional network is the number of atoms to which i belongs, that is, the number of links player i has in g and the number of two-player blocks in atoms below P to which player i belongs.

We now present some properties fulfilled by the lattice of coalitional networks that are of interest for the sequel.

Proposition 3.2 *The lattice $(G\times\mathscr{P},\leq)$ satisfies the JD-condition. The rank function $r:(G\times\mathscr{P})\to\mathbb{N}$ is such that $r(g,P)=n-|P|+|g|$ for all $(g,P)\in G\times\mathscr{P}$.*

Proof. We have $r(g^{\emptyset},P_{\perp})=n-n+0=0$ and for any $(g,P)\in\mathscr{A}(G\times\mathscr{P})$: $r(g,P)=1$ since $r(\ell,P_{\perp})=n-n+1$ and $r(g^{\emptyset},Q_{ij})=n-(n-1)+0$.

Assume that the proposition holds for $(g,P)=(g^N,\{N\})$, i.e. $r(g^N,\{N\})=n-1+\frac{n(n-1)}{2}$. The elements (g,P) covered by $(g^N,\{N\})$ have one of the following two forms : $(g^N\backslash\ell,\{N\})$ or $(g^N,\{N\backslash\{i,j\},\{i,j\}\})$. In the first case, $r(g^N\backslash\ell,\{N\})=n-1+\frac{n(n-1)}{2}-1=r(g^N,\{N\})-1$ and in the second case $r(g^N,\{N\backslash\{i,j\},\{i,j\}\})=n-2+\frac{n(n-1)}{2}=r(g^N,\{N\})-1$. ■

Proposition 3.3 *The lattice $(G\times\mathscr{P},\leq)$ is semimodular.*

Proof. By Proposition 3.2, $(G\times\mathscr{P},\leq)$ is ranked by $r(g,P)=n-|P|+|g|$ for all $(g,P)\in G\times\mathscr{P}$.

Take any $(g,P),(g',P')\in(G\times\mathscr{P})$, then

$$r(g,P)+r(g',P')\geq r(g\cap g',P\wedge P')+r(g\cup g',P\vee P')$$

by first noticing that $|g|+|g'|=|g\cap g'|+|g\cup g'|$ and then $2n-|P|-|P'|\geq 2n-|P\wedge P'|-|P\vee P'|$ by the semimodularity of partition lattices. Conclusion holds by equation (1). ■

4 Coalitional network games

Knowing the lattice structure of coalitional networks in a society, we can now study *games* on coalitional networks that are bottom-normalized real-valued lattice functions :

Definition 4.1 *A (Transferable Utility) coalitional network game is a function* $v : G \times \mathscr{P} \to \mathbb{R}$ *such that* $v(g^{\emptyset}, P_{\perp}) = 0$.

A coalitional network game assigns a real value to each possible pair consisting of a network g and a partition P that represents the total value generated by the society when organized under (g, P). The set of all possible coalitional network games is denoted V.

Note that a coalitional network game is a richer object than a cooperative network game or a characteristic function of a coalitional game as it allows the value generated to depend both on the network structure and on the organization of players into groups. Coalitional network games can be seen as network games with externalities, where the value generated by a network depends on the organization of the society into groups, and converging to classical network games in case of absence of externalities. To see the complexity of coalitional network games, we can compare the possible (vector) space associated to them to the corresponding space of classical network games. Classical network games take values only on the set of possible networks G. The number of possible networks in a society N is $|G| = 2^{\left(\frac{n(n-1)}{2}\right)}$. Network games considered as real-valued function on $|G|$ can be identified with $\mathbb{R}^{|G|}$. The number of possible partitions on a society N is the *Bell*[5] *number* B_n. Considered as real-valued function defined on $G \times \mathscr{P}$, coalitional network games can be identified with $\mathbb{R}^{|G| \times B_n}$.

Definition 4.2 *A coalitional network* $(g, P) \in G \times \mathscr{P}$ *is efficient relative to a coalitional network game* v *if* $v(g, P) \geq v(g', P')$ *for all* $(g', P') \in G \times \mathscr{P}$.

The efficient coalitional network represents the best way to organize the society in terms of network and groups. It represents the coalitional network generating the maximum value.

[5]Bell numbers are defined recursively, using the Stirling numbers of the second kind, and no close form expression exists to express them.

Definition 4.3 *For any coalitional network game $v \in V$, its monotonic cover $\hat{v}$ is defined by*

$$\hat{v}(g,P) = \max_{(g',P') \leq (g,P)} v(g',P').$$

Two different interpretations can be offered to monotonic covers of coalitional network games. The first one corresponds to the one presented by Jackson (2005). The idea is that at the time of building a coalitional network, players consider all the possibilities available, and if there is still some possibility to altering the coalitional network, then it is useful to consider which structure generates the maximum possible value. This approach is called *flexible* by Jackson in the context of network games without externalities.

Another interpretation is the following. In classical coalitional games, it is usually assumed that the grand coalition generates the maximum value and is thus formed. In the coalitional network games context, this is a too strong assumption, as it is often the case that forming or maintaining links induces costs. Instead, we assume here that the complete network and the grand coalition form, but only "*activate*" or "*declare*" some links and groups in order to generate the maximum value. The complete network and the grand coalition have all links and subgroups at their disposal but only use some of them to cooperate. A society with communication links g^N can use any network $g \subset g^N$ to cooperate. A society of players forming a unique group $\{N\}$ are free to group themselves into smaller groups to achieve higher values.[6] Hence, the complete network and the grand coalition always get the maximum value under monotonic cover.

Definition 4.4 *A coalitional network game $v \in V$ is monotonic if*

$$(g,P) \leq (g',P') \Rightarrow v(g,P) \leq v(g',P').$$

Note that if a coalitional network game is monotonic, then $v = \hat{v}$. A monotonic coalitional network game attributes to a coalitional network a higher value than the value it attributes to its subcoalitional networks. This may not be a very natural property in coalitional network games since the top coalitional network structure is not always efficient. Nevertheless, we can draw some useful information about how allocation rules perform on monotonic coalitional network games.

[6] This is the same idea as *essential superadditivity* in coalitional games (see Wooders (2008)).

A special family of monotonic coalitional network games consists in the *unanimity coalitional network games*. For a coalitional network $(g, P) \in G \times \mathscr{P}$, let $u_{g,P} \in V$ denote the coalitional network game satisfying

$$u_{g,P}(g', P') = \begin{cases} 1 & \text{if } (g, P) \leq (g', P') \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We call the coalitional network game $u_{g,P}$ a *unanimity* coalitional network game.

Proposition 4.1 *Let $u_{g,P} \in V$ defined by equation 3. The set*

$$\{u_{g,P} \mid (g, P) \in G \times \mathscr{P}, (g, P) \neq (g^{\emptyset}, P_{\perp})\}$$

of all unanimity coalitional network games forms a linear basis for V.

Proof. This result is a particular case of a general result on lattice functions by Gilboa and Lehrer (1991), Proposition 3.1. ∎

Corollary 4.2 *Each coalitional network game $v \in V$ can be written as*

$$v = \sum_{(g^{\emptyset}, P_{\perp}) \neq (g,P) \in G \times \mathscr{P}} \Delta^{g,P}(v) u_{g,P}. \tag{4}$$

Since $(u_{g,P})_{(g,P) \in G \times \mathscr{P}}$ forms a basis for V, for each $v \in V$, the collection of scalars $(\Delta^{g,P}(v))_{(g,P) \in G \times \mathscr{P}}$ is unique. Each $\Delta^{g,P}(v)$ is called the Harsanyi *dividend* (see Harsanyi (1959)). The dividend of a given element (g, P) of the lattice $(G \times \mathscr{P}, \leq)$ represents the value that is left to (g, P) once all (g', P') below (g, P) have received their corresponding dividends.

Formally, let $v \in V$ and $(g, P) \in G \times \mathscr{P}$. Then

$$v(g, P) = \sum_{(g', P') \leq (g,P)} \Delta^{g',P'}(v) \tag{5}$$

by equations (3) and (4).

By applying the *Möbius inversion formula* (see Rota (1964), Proposition 2), we have

$$\Delta^{g,P}(v) = \sum_{(g', P') \leq (g,P)} v(g', P') \mu\left((g', P'), (g, P)\right) \tag{6}$$

with $\mu\left((g', P'), (g, P)\right)$ the *Möbius function* associated to the lattice $(G \times \mathscr{P}, \leq)$.

Definition 4.5 *The Möbius function associated to the lattice* $(G \times \mathscr{P}, \leq)$ *is defined inductively by*

$$\mu\left((g,P),(g,P)\right) = 1, \qquad \text{for all } (g,P) \in G \times \mathscr{P}$$
$$\mu\left((g_1,P_1),(g_2,P_2)\right) = -\sum_{(g_1,P_1)\leq(g,P)<(g_2,P_2)} \mu\left((g_1,P_1),(g,P)\right)$$
$$\text{for all } (g_1,P_1) < (g_2,P_2) \in G \times \mathscr{P}, \text{ and}$$
$$\mu\left((g_1,P_1),(g_2,P_2)\right) = 0 \qquad \text{otherwise.}$$

We now introduce a general expression for the Möbius function μ associated to the lattice $(G \times \mathscr{P}, \leq)$ instead of the recursive definition 4.5. The two following propositions are useful to determine μ.

We recall that $|P|$ is the number of blocks in partition P. We denote $\mu_{\mathscr{P}}(\cdot,\cdot)$ the Möbius function associated to the partition lattice ordered by refinement.

Proposition 4.3 *(See Schützenberger (1954) and Grabisch (2009)) Let* $P, P' \in \mathscr{P}$ *such that* $P' < P$*. Then the Möbius function on* $(\mathscr{P}, \sqsubseteq)$ *is given by*

$$\mu_{\mathscr{P}}(P',P) = (-1)^{|P'|-|P|}(n_1-1)!\dots(n_{|P|}-1)! \tag{7}$$

where n_k *is the number of blocks of* P' *contained in block* $S_k \in P$*, for each* $k = 1,\dots,|P|$*.*

Denote $\mu_G(.,.)$ the Möbius function associated to the network lattice $(G, \subseteq)$. Remind that $|g|$ is the number of links in network $g \in G$.

Proposition 4.4 *(See Caulier (2009)) Let* $g, g' \in G$ *such that* $g' \subset g$*. Then the Möbius function on* $(G, \subseteq)$ *is given by*

$$\mu_G(g',g) = (-1)^{|g|-|g'|}. \tag{8}$$

Proposition 4.5 *Let* $(g,P),(g',P') \in G \times \mathscr{P}$ *such that* $(g',P') < (g,P)$*. Then the Möbius function on* $(G \times \mathscr{P}, \leq)$ *is given by*

$$\mu\left((g',P'),(g,P)\right) = (-1)^{|P'|-|P|}(n_1-1)!\dots(n_{|P|}-1)!(-1)^{|g|-|g'|} \tag{9}$$

Proof. Direct by $\mu\left((g',P'),(g,P)\right) = \mu_G(g',g)\mu_{\mathscr{P}}(P',P)$ and equations (7) and (8). ■

By equation (4), we know that the unanimity coefficients of any coalitional network game are given by the Harsanyi dividends, i.e. the Möbius inversion formula. In order to know the value taken by each dividend, we only have to plug equation (9) into equation (6):

$$\Delta^{g,P}(v) = \sum_{(g',P')\leq(g,P)} v(g',P')(-1)^{|P'|-|P|}(n_1-1)!\dots(n_{|P|}-1)!(-1)^{|g|-|g'|}.$$

Another way to get the value of dividends is by recurrence :

$$\Delta^{(g^{\emptyset},P_{\perp})}(v) = 0$$

and using equation (5) :

$$\Delta^{(g,P)}(v) = v(g,P) - \sum_{(g',P')<(g,P)} \Delta^{(g',P')}(v). \tag{10}$$

By equation (10) we see clearly the interpretation of a dividend: the dividend of a coalitional network (g,P) is the part of the value $v(g,P)$ that is not generated by proper subcoalitional networks of (g,P).

5 Flexible coalitional networks and equal treatment

In this section, we propose an allocation rule for flexible coalitional networks. First we provide the formal definition of an allocation rule for coalitional network games.

Definition 5.1 *An allocation rule for a coalitional network game $v \in V$ is a function $\psi : G \times \mathscr{P} \times V \to \mathbb{R}^N$ such that $\sum_i \psi_i(g,P,v) = v(g,P)$ for all v, g and P.*

An allocation rule specifies how the value generated by a coalitional network is allocated among players. Note that balance ($\sum_i \psi_i(g,P,v) = v(g,P)$) is included in the definition and that an allocation rule depends on g, P and v.

Definition 5.2 *An allocation rule ψ is a flexible coalitional network rule if*

$$\psi\left(g,P,v\right) = \psi\left(g^N,\{N\},\hat{v}\right)$$

for all v and efficient coalitional network (g,P) relative to v.

The allocation rule only depends on the monotonic cover of the coalitional network game and distributes the value taken by the efficient configuration. This is consistent with the perspective that the coalitional network is being formed or that it can still be modified or that it can be declared as another configuration reachable. The idea from the flexible perspective is that inefficient coalitional network structures should not be reached.

Note in the definition that the equivalence is only required on efficient structures, as the value that accrues to other coalitional networks might not even be the same (i.e. $v(g, P) \neq \hat{v}(g, P)$ for inefficient (g, P)).

The next property is a kind of separable consistency. The property states the behavior followed by the players concerning the repartition of the value generated when their society is confronted to two different games.

Definition 5.3 *An allocation rule ψ is weakly additive if for any monotonic coalitional nework games v and v', and scalars $a \geq 0$ and $b \geq 0$,*

$$\psi(g^N, \{N\}, av + bv') = a\psi(g^N, \{N\}, v) + b\psi(g^N, \{N\}, v'),$$

and if $av - bv'$ is monotonic, then

$$\psi(g^N, \{N\}, av - bv') = a\psi(g^N, \{N\}, v) - b\psi(g^N, \{N\}, v').$$

Again here, the weakly additivity condition only applies to monotonic covers, the only relevant information if we consider the coalitional network as flexible.

As a matter of equity, Jackson (2005) proposes to share the value in a unanimity game equally between essential players or, for link-based allocation rules, between essential links, whichever you consider as "vital" in generating value. In coalitional network games, basic ingredients are not the players. The mathematical structure in terms of lattice shows that the minimal aggregation form in a coalitional network is an atom, which takes the form of either a link between two players or a partition whose unique non-singleton block is a pair of players. In order to assess the contribution to cooperation of players in this context, we argue that the role played by each atom must first be assessed. We now develop formally this argument.

Let $P \in \mathscr{P}$ be a k-partition $\{S_1, \ldots, S_k\}$ and $A \subset N$ a nonempty subset of players. We denote the *restriction* of P to A by $P_{|A} := \{S_1 \cap A, \ldots, S_k \cap A\}$. For any coalitional network game $v \in V$, the *restricted coalitional network game* $v_{|A}$ is defined by

$$v_{|A}(g,P) := v\left(g_{|A}, \{P_{|A}\} \cup \{P^{\perp}_{|N\setminus A}\}\right).$$

The restricted coalitional network game is thus the game defined on a proper sub-coalitional network determined by the coalition $A \subset N$, with players in A partitioned according to P and players in $N \setminus A$ being singletons.

To see the importance of atoms in the production of value, let $v \in V$, $(ij, P_{\perp}) \in \mathscr{A}(G \times \mathscr{P})$. Then

$$\begin{aligned} v(ij,P_{\perp}) - v_{|N\setminus\{i\}}(ij,P_{\perp}) =& v(ij,P_{\perp}) - v_{|N\setminus\{j\}}(ij,P_{\perp}) \\ =& v(ij,P_{\perp}) - v_{|N\setminus\{ij\}}(ij,P_{\perp}) \\ =& v(ij,P_{\perp}). \end{aligned}$$

Similarly, let $(g^{\emptyset}, Q_{ij}) \in \mathscr{A}(G \times \mathscr{P})$. Then

$$\begin{aligned} v(g^{\emptyset},Q_{ij}) - v_{|N\setminus\{i\}}(g^{\emptyset},Q_{ij}) =& v(g^{\emptyset},Q_{ij}) - v_{|N\setminus\{j\}}(g^{\emptyset},Q_{ij}) \\ =& v(g^{\emptyset},Q_{ij}). \end{aligned}$$

In the first case, the value is generated by the link ij and players i and j have a symmetric role in this link. In the second case, the value is due to the presence of the group $\{i,j\}$ and players i and j have a symmetric role in this group. The argument is also valid for any (g,P) such that in the first case g is a collection of disjoint links or in the second case P is a collection of atoms whose pairs are disjoint.

Now let any $(g,P_{\perp}) \in G \times \mathscr{P}$ and $v \in V$. Then,

$$\begin{aligned} v_{|N\setminus\{i\}}(g,P_{\perp}) =& v\left(g_{|N\setminus\{i\}}, P_{\perp}\right) \\ =& v\left(g \setminus N_i(g), P_{\perp}\right) \end{aligned}$$

with $N_i(g)$ the set of links involving player i. This means that the value of a coalitional network in a society excluding player i and such that $P = P_{\perp}$, is equivalent to the value of the coalitional network after removing all atoms $(\ell, P_{\perp})$ such that $i \in \ell$.

Similarly, take any $(g^{\emptyset}, P) \in G \times \mathscr{P}$ and $v \in V$. The decomposition in terms of atoms of P is $P = \bigvee_{k \in K} Q_k$ with $Q_k \in \mathscr{A}(\mathscr{P})$ for each $k \in K$, K a set of indices. Let K_{-i} the set of indices of atoms such that i is not a member of the only pair in the atom, i.e. the set of atoms in the decomposition of P with player i singleton.

Then,

$$v_{|N\setminus\{i\}}(g^{\emptyset}, P) = v\left(g^{\emptyset}, P_{|N\setminus\{i\}}\right) = v\left(g^{\emptyset}, \bigvee_{k\in K_{-i}} Q_k\right).$$

Again, the value of a coalitional network in a society excluding a player i and such that $g = g^{\emptyset}$, is equivalent to the value of the coalitional network after removing all atoms in the decomposition of P in which player i is not singleton.

The total contribution of a player i in a given coalitional network is thus the sum of the contributions of the different atoms to which i is a member. For this reason, in order to properly analyze the production of value and its allocation, we have to analyze the basic ingredients generating a value. The atoms of the lattice of coalitional networks are these minimal aggregation ingredients.

Hence we propose the following property:

Definition 5.4 *An allocation rule ψ satisfies equal treatment of atoms if $u_{g,P} \in V$ is a unanimity coalitional network game for some coalitional network (g, P), then*

$$\psi_i(u_{g,P}) = \begin{cases} \sum_{\substack{(g_a,P_a)\in\mathscr{A}(g,P),\\ i\in(g_a,P_a)}} \frac{1}{2|\mathscr{A}(g,P)|} & \text{if } i \text{ belongs to at least one atom,} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that unanimity coalitional network games are the ones where the atoms below (g, P) are all members of the decomposition of (g, P), i.e. whose join is the (only) configuration that generates some value. Formally, for each $(g, P) \in G \times \mathscr{P}$ with $\mathscr{A}(g, P) \subset \mathscr{A}(G \times \mathscr{P})$ the set of atoms such that $(g_a, P_a) \in \mathscr{A}(g, P) \Rightarrow (g_a, P_a) \leq (g, P)$, we have $(g, P) = \bigcup_{(g_a,P_a)\in\mathscr{A}(g,P)} (g_a, P_a)$. In a unanimity game $u_{g,P}$, atoms below (g, P) are all in some sense "equals" since all other atoms contribute nothing, they are not part of the structure generating the value. We thus consider as natural to distribute the value equally among these atoms.

The properties described above are enough to characterize a unique solution, that we call the *atom-based flexible coalitional network allocation rule*:

Theorem 5.1 *An allocation rule for coalitional network games satisfies equal treatment of atoms, weak additivity and is a flexible coalitional network rule if and only* *if for all $v \in V$ and $(g,P) \in G \times \mathscr{P}$ efficient relative to v, the atom-based flexible coalitional network allocation rule ψ is defined by*

$$\psi_i(g,P,v) = \sum_{(g,P)\in G\times\mathscr{P}} \sum_{\substack{(g_a,P_a)\in\mathscr{A}(g,P):\\ i\in(g_a,P_a)}} \frac{\Delta^{g,P}(\hat{v})}{2|\mathscr{A}(g,P)|} \tag{11}$$

The idea is first to calculate the dividends for the monotonic cover of the game under consideration, next, to distribute these dividends equally among the atoms below the coalitional networks corresponding to the dividends and, finally, to the players essential to these atoms.

Proof. First we show that equation (11) satisfies all the properties. Observe that, by equation (10),

$$\sum_{(g,P)} \Delta^{g,P}(\hat{v}) = \hat{v}(g^N,\{N\}) = max_{(g,P)\in G\times\mathscr{P}} v(g,P)$$

and thus

$$\sum_{i\in N} \psi_i(g,P,v) = \hat{v}(g^N,\{N\})$$

since each atom consists in two players.

Equation (11) satisfies weak-additivity: Consider any monotonic coalitional network games v and v' in V, and scalars $a \geq 0$ and $b \geq 0$. Then $av + bv'$ is monotonic and coincides with its monotonic cover. Hence,

$$\begin{aligned}\psi_i(g^N,\{N\},av+bv') &= \sum_{(g,P)\in G\times\mathscr{P}} \sum_{\substack{(g_a,P_a)\in\mathscr{A}(g,P)\\ i\in(g_a,P_a)}} \frac{\Delta^{g,P}(a\hat{v}+b\hat{v}')}{2|\mathscr{A}(g,P)|}\\ &= \sum_{(g,P)\in G\times\mathscr{P}} \sum_{\substack{(g_a,P_a)\in\mathscr{A}(g,P)\\ i\in(g_a,P_a)}} \frac{a\Delta^{g,P}(\hat{v})+b\Delta^{g,P}(\hat{v}')}{2|\mathscr{A}(g,P)|}\\ &= a\psi(g^N,\{N\},v) + b\psi(g^N,\{N\},v').\end{aligned}$$

Where the second equality holds by equation (10).

By a similar argument if $av - bv'$ is monotonic, we show that $\psi_i(av - bv') = a\psi_i(v) - b\psi_i(v')$.

Equal treatment of atoms is easily checked to hold in equation (11).

Next, let us verify that any allocation rule satisfying equal treatment of atoms, weak additivity, and flexible coalitional network must coincide with the atom-based flexible coalitional network allocation rule ψ on efficient coalitional networks.

Let $v \in V$ and $\phi : G \times \mathscr{P} \times V \to \mathbb{R}^N$ an allocation rule satisfying the claimed properties. Given that ϕ is a flexible coalitional network allocation rule implies that $\phi(g, P, v) = \phi(g^N, \{N\}, \hat{v})$ on efficient (g, P) relative to v, and so it is enough to show that $\phi(g^N, \{N\}, \hat{v})$ is uniquely determined on an efficient coalitional network.

By Corollary 4.2,

$$\hat{v} = \left(\sum_{(g,P) \in G \times \mathscr{P}} \Delta^{g,P}(\hat{v}) u_{g,P} \right)$$

Let $G^- = \left\{ (g, P) \mid \Delta^{g,P} < 0 \right\}$ and $G^+ = G \times \mathscr{P} \setminus G^-$. Hence,

$$\hat{v} = \sum_{(g,P) \in G^+} \Delta^{g,P}(\hat{v}) u_{g,P} - \sum_{(g,P) \in G^-} \left| \Delta^{g,P}(\hat{v}) \right| u_{g,P}.$$

By weak additivity,

$$\phi\left(g^N, \{N\}, \hat{v}\right) = \phi\left(g^N, \{N\}, \sum_{(g,P) \in G^+} \Delta^{g,P}(\hat{v}) u_{g,P}\right) - \phi\left(g^N, \{N\}, \sum_{(g,P) \in G^-} \left| \Delta^{g,P}(\hat{v}) \right| u_{g,P}\right)$$

By weak additivity again,

$$\phi\left(g^N, \{N\}, \hat{v}\right) = \sum_{(g,P) \in G \times \mathscr{P}} \Delta^{g,P}(\hat{v}) \phi(g^N, \{N\}, u_{g,P}).$$

Since ϕ is a flexible coalitional network allocation rule then $(g^N, \{N\})$ and (g, P) take both the same value under the monotonic cover of $u_{g,P}$ for each $(g, P) \in G \times \mathscr{P}$. Finally, by equal treatment of atoms, the value is uniquely determined and thus, $\phi = \psi$. ■

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