Implementation and revenue equivalence without differentiability*

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April 2010

Abstract

We introduce a characterization of (dominant strategy) implementable allocation rules based on an integral monotonicity condition. This condition relates valuation differences with the integral of measurable selections of the subderivative correspondence between two types, defined at equilibrium allocations. We use this characterization, which does not rely on convexity or full differentiability assumptions of the valuation function with respect to types, to provide a generalized Revenue Equivalence result that holds even when the standard version fails. Our new version of Revenue Equivalence imposes bounds on the difference between indirect utility functions generated by two payment schemes that implement the same allocation rule and assign the same equilibrium payoff to the "lowest type". We provide some examples to illustrate our results.

Keywords: Implementation, Revenue Equivalence, Integral Monotonicity, Subderivative Correspondence, Integral of a Correspondence.

JEL Classification Numbers: C72, D70, D82, D44.

1 Introduction

The Revenue Equivalence Principle states that, under certain conditions, two (dominant strategy) incentive compatible mechanisms sharing the same allocation rule generate revenue that differ at most by a constant. The derivation of this remarkable result was originally accomplished by Myerson (1981) under the assumptions that types lie in one-dimensional intervals and valuations are linear in types. The technique Myerson used,

*Preliminary version; comments welcome. The first author is indebted to Claudio Mezzetti and Rabee Tourky for helpful discussions on the subject.

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i.e., imposing sufficient conditions on different aspects of the design problem to apply the envelope theorem, was later adapted to prove more general versions of Revenue Equivalence.[1] For instance, Milgrom and Segal (2002) worked with an arbitrary set of social allocations and obtained the integral representation of the value function of the individual incentive problem under the hypotheses that valuations are absolutely continuous and everywhere differentiable with respect to one-dimensional types.[2] More recently, Berger, Müller, and Naeemi (2009) use their characterization of implementable allocation rules, based on well-known monotonicity and integrability conditions (in addition to their *decomposition monotonicity* assumption), to show that Revenue Equivalence also holds when the type space is a convex subset of a multi-dimensional Euclidean space and the valuation function is convex in types.

In this paper, we explore the consequences for Revenue Equivalence and (dominant strategy) implementation of dropping the assumptions of differentiability and convexity of the valuation function with respect to types. These assumptions, suitable as they may be in several economic situations, are not present in a variety of interesting problems. Moreover, their absence compels us to introduce new techniques that may in turn be valuable in economic settings lacking smooth or convex valuation functions. As an illustration, consider the following example.

Example 1. The set of alternatives is $\mathcal{X} = (0,1)$. There is a single agent with a quasi-linear utility function $u = v(x,\theta) - \rho$ defined over alternatives $x \in \mathcal{X}$ and monetary payments $\rho \in \mathbb{R}$. Types, which are private information, lie in $\Theta = (0,1)$. The agent's valuation function $v\colon \mathcal{X} \times \Theta \to \mathbb{R}$ is given by $v(x,\theta) = -|x-\theta|$. We think of $\mathcal{X} = (0,1)$ as the set of possible locations for a library. Our agent resides at location $\theta \in (0,1)$ and pays a linear cost to travel to the library, were it not located at θ. Note that for every location x, the function $\theta \mapsto v(x,\theta)$ is neither convex nor fully differentiable on Θ; in particular, its derivative with respect to θ fails to exist whenever $\theta = x$.

The efficient allocation rule $X^*\colon \Theta \to \mathcal{X}$ selects $X^*(\theta) = \theta$. Clearly X^* is implementable, although the characterization theorem of Berger, Müller, and Naeemi (2009) is moot, since the valuation function lacks convexity with respect to types. On the other hand, the agent's valuation is never differentiable with respect to types at the equilibrium points $(X^*(\theta),\theta)$. Thus, although Milgrom and Segal (2002, Theorem 2) show that, for any payment rule $p\colon \Theta \to \mathbb{R}$ implementing X^*, the indirect utility function $\theta \mapsto U(\theta) \equiv v(X^*(\theta),\theta) - p(\theta)$ is absolutely continuous and thus equal to the integral of its derivative (which exists almost everywhere), their envelope theorem implies nothing more than

$$-1 \;=\; \overline{\mathrm{d}}v(X^*(\theta),\theta) \;\leq\; DU(\theta) \;\leq\; \underline{\mathrm{d}}v(X^*(\theta),\theta) \;=\; 1,$$

[1] Notable recent exceptions to this approach are Heydenreich, Müller, Uetz, and Vohra (2009) and Chung and Olszewski (2007).

[2] Krishna and Maenner (2001) obtained a similar result with weaker assumptions on the valuation functions, at the cost of restricting the set of social alternatives and requiring the mechanisms to satisfy certain differentiability property.

where $\overline{\mathrm{d}}v(x,\cdot)$ denotes the right-hand partial derivative of $v(x,\cdot)$ with respect to θ and $\underline{\mathrm{d}}v(x,\cdot)$ denotes its left-hand counterpart. Thus, a priori, incentive compatibility of the direct mechanism (X^*,p) implies nothing more than

$$U(\theta_1) \;-\; U(\theta_0) \;=\; \int_{\theta_0}^{\theta_1} s(\theta)\,d\theta, \qquad\qquad 0<\theta_0,\theta_1<1; \tag{1}$$

for *some* integrable selection s from the correspondence $S\colon \Theta \rightrightarrows \mathbb{R}$ defined by

$$S(\theta) \;=\; \big[\,\overline{\mathrm{d}}v(X^*(\theta),\theta), \underline{\mathrm{d}}v(X^*(\theta),\theta)\,\big] \;=\; [-1,1].$$

This contrasts with the standard version of Revenue Equivalence, which relies on the full differentiability or convexity of $v(X^*(\theta),\cdot)$ with respect to types to yield (a unique) $Dv(X^*(\theta),\theta)$ almost everywhere in Θ to take the place of the integrand in equation (1).[3] One cannot conclude that the indirect utility is determined solely by X^*, since there is the possibility that the selection s depends also on the payment rule. We claim that this is the case.

Indeed, note X^* is implementable by a constant payment rule $p \equiv 0$. The indirect utility function is in this case $U(\theta)=0$ with $DU(\theta)=0$, for all $\theta\in\Theta$. Now consider the alternative payment rule $\overline{p}$ defined on Θ by $\overline{p}(\theta)=\theta$. To see that $(X^*,\overline{p})$ is incentive compatible, fix a type $\theta_0\in\Theta$. Reporting θ_1 gives our agent payoffs of the form:

$$v(X^*(\theta_1),\theta_0) \;-\; \overline{p}(\theta_1) \;=\; \begin{cases} -\,\theta_0, & \text{if } \theta_1\le\theta_0; \\ \theta_0 \;-\; 2\theta_1, & \text{if } \theta_1>\theta_0. \end{cases}$$

It follows that truth-telling is an equilibrium strategy. The indirect utility function $\overline{U}$ generated by $\overline{p}$ satisfies $\overline{U}(\theta_1)-\overline{U}(\theta_0) \;=\; -\,(\theta_1-\theta_0)$, thus equation (1) holds for the selection $\overline{s}(\theta)=-\,1$.

A similar argument shows that the payment rule $\underline{p}$, defined on Θ by $\underline{p}(\theta) \;=\; -\,\theta$, also implements X^* and generates an indirect utility satisfying $\underline{U}(\theta_1)-\underline{U}(\theta_0)=\theta_1-\theta_0$, with equation (1) valid for the selection $\underline{s}(\theta)=1$. In this example, in particular, one can show that for *any* integrable selection from the correspondence S, there exists a payment rule that implements X^* and generates an indirect utility function consistent with expression (1). Take the selection $s^*(\theta)=\theta$, all $\theta\in\Theta$. It can be verified that the payment rule p^* such that $p^*(\theta) = -\,\frac{1}{2}\theta^2$, all $\theta\in\Theta$, implements X^* and generates an indirect utility function for which $U^*(\theta_1)-U^*(\theta_0) \;=\; -\,\frac{1}{2}(\theta_1^2-\theta_0^2)$. $\square$

This example clearly illustrates how Revenue Equivalence is lost when the valuation function fails to be differentiable or convex with respect to types. Note, however, that for each allocation $x\in\mathcal{X}=(0,1)$, the function $\theta\mapsto v(x,\theta)$ is Lipschitz on $\Theta=(0,1)$ with

[3] In addition to Milgrom and Segal (2002), see Berger, Müller, and Naeemi (2009), Krishna and Maenner (2001) and Williams (1999).

Lipschitz constant $\ell(x) = 1$. We shall maintain these features in our general environment, presented in Section 2. More specifically, we deal with a single agent with quasi-linear preferences over alternatives x and money (extensions of the model and results to multi-agent environment are straightforward). We assume that (A1) the allocation set $\mathcal{X}$ is a measurable space; (A2) the type space $\Theta \subseteq \mathbb{R}^k$ is open, convex and bounded (for $k \geq 1$); (A3) for each x, the function $v(x, \cdot)$ is Lipschitz continuous on Θ, with bounded Lipschitz constants. No further assumption is made on the primitives of the design model. We notice that the cases of valuation functions that are linear, convex, or differentiable in types are covered by our model (with the addition of an appropriate boundedness condition).

Our assumption (A3) provides us with upper and lower subderivatives of the valuation function with respect to types, which are then used to construct the subderivative correspondence between any two types θ_0, θ_1 in Θ.[4] In Section 3, we impose (M) a measurability requirement on the subderivatives of the valuation with respect to types evaluated at equilibrium points. We show that if the subderivative functions generated by an implementable allocation rule are consistent with (M), the subderivative correspondence between θ_0 and θ_1 is non empty-valued a.e. in the line segment connecting θ_0 to θ_1, closed-valued and measurable, with its integral being a non-empty, closed convex subset of $\mathbb{R}$. Our characterization result resorts to these properties (see Theorem 4): an allocation rule is implementable if and only if the subderivative correspondence between any two types admits an integrable selection for which the integral monotonicity condition and the path-integrability condition are satisfied.

It comes as no surprise that a monotonicity condition is present in our characterization result. Since the seminal work of Rochet (1987), who characterized implementation via cyclic monotonicity, work in mechanism design has been devoted to the study of situations where the weaker 2-cycle monotonicity is not only necessary but also sufficient for implementation. Saks and Yu (2005) showed that this holds if the allocation set is finite and the type space is convex (Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006) proved a similar result for auction-like environments). More recently, Archer and Kleinberg (2008) and Berger, Müller, and Naeemi (2009) extended the characterization of implementable allocation rules via weak monotonicity (plus a path integrability condition) to environments with arbitrary allocation sets and multi-dimensional convex type spaces; the former under the assumption that valuations are linear in types, while the latter assumed convex valuations instead. For our characterization theorem, we dispense of both linearity and convexity assumptions. As a consequence, we work with (possibly many) integral selections of the subderivative correspondence and state the monotonicity and the path-integrability conditions in terms of these selections. At the end of Section 3 we relate our result and our assumption (M) to previous work.

Our characterization result does not imply that two incentive compatible mechanisms with the same allocation rule generate equilibrium payoffs (hence revenue) that differ at

[4] Mathematical concepts and results employed in this paper are presented at the end of Section 2.

most by a constant (see Example 1 above). Hence the standard version of the Revenue Equivalence Principle fails. However, as we demonstrate in Section 4, Theorem 4 implies that the difference in equilibrium payoffs generated by two incentive compatible mechanisms with the same allocation rule is bounded, with the bounds depending solely on the allocation rule. Moreover, we are able to show that if two payment schemes implement the same allocation rule and generate indirect utility functions U and U', respectively (where U' is not an affine translation of U), then for every convex combination of U and U' there exists a payment scheme that implements the allocation rule and generates an indirect utility equal to $\lambda U + (1-\lambda)U'$. We end Section 4 with a sufficient condition on the subderivative correspondence to obtain the standard version of Revenue Equivalence. This condition is satisfied whenever the valuation function is linear, convex or differentiable in types.

2 Preliminaries

2.1 The economic environment

We consider a mechanism design setting with only one agent; extensions of our model and results to multi-agent settings are immediate. An outcome is a pair (x, ρ), where x represents a social alternative from the *allocation set* $\mathcal{X}$ and the real number ρ represents some quantity of a perfectly divisible, private commodity (money). Our agent has *quasi-linear preferences* over outcomes, so that

$$u(x, \theta, \rho) \;=\; v(x, \theta) \;-\; \rho$$

represents the agent's utility when allocation $x \in \mathcal{X}$ is selected and the amount $\rho \in \mathbb{R}$ is paid by her, given that her privately known type is θ. We denote the agent's *type space* by Θ and refer to $v \colon \mathcal{X} \times \Theta \to \mathbb{R}$ as the agent's *valuation function*.

The following assumptions are imposed on our model.

(A1) The pair $(\mathcal{X}, \boldsymbol{\mathcal{X}})$ is a measurable space ($\boldsymbol{\mathcal{X}}$ denotes a σ-algebra of subsets of $\mathcal{X}$).

(A2) The type space Θ is an open, convex, bounded subset of $\mathbb{R}^k$ ($k \geq 1$).

(A3) For every $x \in \mathcal{X}$, the function $\theta \mapsto v(x, \theta)$ is Lipschitz continuous on Θ; i.e., there exists a positive real number $\ell(x)$ such that

$$|v(x, \theta) - v(x, \hat{\theta})| \;\leq\; \ell(x)\, \|\theta - \hat{\theta}\|,$$

for all $\theta, \hat{\theta} \in \Theta$. Furthermore, the set of real numbers $\{\ell(x) \colon x \in \mathcal{X}\}$ is bounded above, with $\ell = \sup\{\ell(x) \colon x \in \mathcal{X}\} < +\infty$.

(A1) does not impose any burdensome limitation on the allocation set, which is allowed to be finite or infinite. The convexity of the type space is a standard assumption in the mechanism design literature with parametric valuations, and is satisfied in several economic applications. (A2) considers an open convex $\Theta \subseteq \mathbb{R}^k$, but this could be relaxed to include convex sets with non-empty interiors. The boundedness of the type space plays a technical role in some our results. Assumption (A3) provides the existence of the subderivative of the valuation function with respect to types, a weaker concept than that of the standard directional derivative. No further restrictions are imposed a priori on the primitives of our environment. In particular, in contrast to previous work in this topic, we do not assume linearity, convexity, or full differentiability of the valuation function with respect to types. Of course any such extra assumption, with the addition of an appropriate boundedness condition, suffices to obtain (A3). Observe that Example 1 satisfies (A1) to (A3).

The allocation problem is resolved via direct mechanisms of the form (X, p). The function $X\colon \Theta \to \mathcal{X}$ is called an *allocation rule* and the function $p\colon \Theta \to \mathbb{R}$ is called a *payment rule*. We say that X is *implementable* if there exists a payment rule p such that truth-telling is an equilibrium strategy for our agent; i.e.,

$$v(X(\theta),\theta) - p(\theta) \;\geq\; v(X(\hat{\theta}),\theta) - p(\hat{\theta}), \qquad \text{all } \theta,\hat{\theta} \in \Theta. \tag{2}$$

In such situation, the direct mechanism (X, p) is said to be *incentive compatible* and the function $U\colon \Theta \to \mathbb{R}$ defined by

$$U(\theta) \;\equiv\; v(X(\theta),\theta) - p(\theta), \qquad \text{all } \theta \in \Theta, \tag{3}$$

is called the agent's *indirect utility generated by* (X, p). We shall restrict our analysis to measurable allocation rules. Extensions of the notions of (dominant strategy) implementable allocation rules and (dominant strategy) incentive compatible mechanisms to multi-agent settings are readily obtained.

2.2 Subderivatives and the integral of a correspondence

We introduce here concepts and results that are used in our characterization theorem. For details, the reader is referred to Aubin and Frankowska (1990), Hildenbrand (1974) and Rockafellar and Wets (1998).

Fix an allocation $x \in \mathcal{X}$ and a vector $\delta \in \mathbb{R}^k, \delta \neq \mathbf{0}$. The right and left subderivatives of the function $\theta \mapsto v(x,\theta)$ evaluated at $\hat{\theta} \in \Theta$ in the direction δ are defined as the following lower and upper limits:

$$\overline{\mathrm{d}}v(x,\hat{\theta})(\delta) \;:=\; \liminf_{r\downarrow 0} \frac{v(x,\hat{\theta}+r\delta) - v(x,\hat{\theta})}{r}; \tag{4}$$

$$\underline{\mathrm{d}}v(x,\hat{\theta})(\delta) \;:=\; \limsup_{r\uparrow 0} \frac{v(x,\hat{\theta}+r\delta) - v(x,\hat{\theta})}{r}. \tag{5}$$

The Lipschitz continuity of $v(x,\cdot)$ in (A3) implies that these subderivatives exist and are finite for every $\hat{\theta} \in \Theta$, for all $x \in \mathcal{X}$ and $\delta \in \mathbb{R}^k$. Notice that $\underline{\mathrm{d}}v(x,\hat{\theta})(\delta) = -\,\overline{\mathrm{d}}v(x,\hat{\theta})(-\,\delta)$. Clearly, if $\theta \mapsto v(x,\theta)$ admits one-sided directional derivatives on Θ, then we can replace the upper and lower limits in (4) and (5) with the usual one-sided limits, although the above notation shall be maintained. Given types θ_0, θ_1 in Θ, denote their vector difference by $\delta_0^1 = \theta_1 - \theta_0 \in \mathbb{R}^k$. The line segment connecting θ_0 to θ_1 is the set $L(\theta_0,\theta_1) = \{\, \theta_0 + \alpha\,\delta_0^1 \mid \alpha \in [0,1]\,\}$. By (A2), one has $L(\theta_0,\theta_1) \subseteq \Theta$ for all $\theta_0, \theta_1 \in \Theta$. We shall consider the function $\alpha \mapsto \theta_0^1(\alpha) = \theta_0 + \alpha\,\delta_0^1$ mapping $[0,1]$ onto $L(\theta_0,\theta_1)$.

Let $\mathcal{B}([0,1])$ denote the Borel σ-algebra of subsets of $[0,1]$. Let $S\colon [0,1] \rightrightarrows \mathbb{R}$ be a correspondence with closed images. Then S is said to be a measurable correspondence if for every open set O in $\mathbb{R}$, the inverse image $S^{-1}(O) = \{\, \alpha \in [0,1] \mid S(\alpha) \cap O \neq \emptyset \,\}$ belongs to $\mathcal{B}([0,1])$; in particular, $\operatorname{dom} S = \{\, \alpha \in [0,1] \mid S(\alpha) \neq \emptyset \,\}$ and its complement are measurable sets. Further, S is said to integrably bounded if there exists a non-negative (Lebesgue) integrable function g defined on $[0,1]$ such that $S(\alpha) \subseteq [-\,g(\alpha), g(\alpha)]$ for almost all α in $[0,1]$ (with respect to the Lebesgue measure). A selection s of the correspondence S is a function $\alpha \mapsto s(\alpha)$ such that $s(\alpha) \in S(\alpha)$, for almost all $\alpha \in [0,1]$. By the Measurable Selection Theorem, a measurable correspondence $S\colon [0,1] \rightrightarrows \mathbb{R}$ admits a measurable selection $s\colon [0,1] \to \mathbb{R}$. If in addition S is integrably bounded, then it admits an integrable selection. In such case, the integral of the correspondence S is the non-empty set of real numbers

$$\Big\{ \int_0^1 s(\alpha)\, d\alpha \mid s(\alpha) \in S(\alpha) \text{ a.e. in } [0,1] \Big\}.$$

By the Lyapunov's Convexity Theorem, the integral of a closed-valued, measurable and integrably bounded correspondence $S\colon [0,1] \rightrightarrows \mathbb{R}$ is a non-empty closed interval.

3 Characterizing implementable allocation rules

Consider a direct mechanism (X,p). It is not difficult to realize that, whenever (X,p) is incentive compatible, the Lipschitz properties of the valuation function in (A3) imply the Lipschitz continuity of the indirect utility function U generated by (X,p). Furthermore, as our first lemma shows, U has two-sided directional derivatives almost every in the line segment connecting any two types.

Lemma 2. *Assume that $p\colon \Theta \to \mathbb{R}$ implements the allocation rule X.*

(a) The indirect utility function U generated by (X,p) is Lipschitz continuous on Θ.

(b) For every pair θ_0, θ_1 of distinct types in Θ, U admits two-sided directional derivatives in the direction $\delta_0^1 = \theta_1 - \theta_0$ a.e. in the line segment $L(\theta_0,\theta_1)$.

Proof (a) Consider θ_0, θ_1 in Θ. From expression (3) one sees that

$$\begin{aligned} U(\theta_1) - U(\theta_0) &\leq \big\{v(X(\theta_1), \theta_1) - p(\theta_1)\big\} \;-\; \big\{v(X(\theta_1), \theta_0) - p(\theta_1)\big\} \\ &= v(X(\theta_1), \theta_1) \;-\; v(X(\theta_1), \theta_0) \\ &\leq \ell(X(\theta_1))\, \|\theta_1 - \theta_0\| \;\leq\; \ell\, \|\theta_1 - \theta_0\|; \end{aligned}$$

where the last two inequalities follow from (A3). Reversing the roles of θ_0 and θ_1, one readily concludes that $|U(\theta_1) - U(\theta_0)| \leq \ell \|\theta_1 - \theta_0\|$. Since θ_0 and θ_1 were arbitrarily chosen, this shows that U is Lipschitz on Θ.

(b) Given distinct types θ_0, θ_1 in Θ, let $\theta_0^1(\alpha) = \theta_0 + \alpha\, \delta_0^1$ belong to the line segment $L(\theta_0, \theta_1)$, for $\alpha \in [0, 1]$. Define the real-valued function μ on $[0, 1]$ by $\mu(\alpha) = U(\theta_0^1(\alpha))$. It is readily seen that μ is Lipschitz continuous. Indeed, for any α, α' in $[0, 1]$ one has:

$$\begin{aligned} |\mu(\alpha) - \mu(\alpha')| &= |U(\theta_0^1(\alpha)) - U(\theta_0^1(\alpha'))| \\ &\leq \ell\, \|\theta_0^1(\alpha) - \theta_0^1(\alpha')\| \;=\; \ell\, \|\delta_0^1\|\, |\alpha - \alpha'|, \end{aligned}$$

where the inequality in the above expression follows from part (a) of the lemma. Thus, μ is Lipschitz on $[0, 1]$ and therefore absolutely continuous and differentiable a.e. in $[0, 1]$. In particular, if μ is differentiable at $\alpha \in (0, 1)$, then we deduce:

$$\begin{aligned} D\mu(\alpha) &= \lim_{r \to 0} \frac{\mu(\alpha + r) \;-\; \mu(\alpha)}{r} \\ &= \lim_{r \to 0} \frac{U(\theta_0^1(\alpha + r)) \;-\; U(\theta_0^1(\alpha))}{r} \\ &= \lim_{r \to 0} \frac{U(\theta_0^1(\alpha) + r\, \delta_0^1) \;-\; U(\theta_0^1(\alpha))}{r} \;=\; DU(\theta_0^1(\alpha))(\delta_0^1), \end{aligned}$$

where the right-hand side of the last equality denotes the two-sided directional derivative of U evaluated at $\theta_0^1(\alpha)$ in the direction δ_0^1. The result now follows. ■

Milgrom and Segal (2002) obtained an analogue of Lemma 2(a) under the alternative hypothesis of absolute continuity of $v(x, \cdot)$ on a one-dimensional type space, plus an integral bound condition on the derivative of v with respect to types. Any Lipschitz function defined on an interval is absolutely continuous. However, the extension of the absolute continuity concept to multi-dimensional settings is not straightforward.[5] On the other hand, the Lipschitz continuity property extends naturally to multi-dimensional domains and allows us to work with upper and lower subderivatives of the function $\theta \mapsto v(x, \theta)$ everywhere in the type space.

Observe that Lemma 2(b) does not imply that U is fully differentiable a.e. in every curve connecting θ_0 and θ_1. In fact, there may be many (piecewise) smooth curves

[5] For instance, it is possible to construct a convex function on a plane that fails to be absolutely continuous; see Friedman (1940) for details. Such function, of course, is Lipschitz on any compact set contained in the interior of its domain.

joining θ_0 to θ_1 for which the indirect utility function U is nowhere fully differentiable in such curves.[6] What Lemma 2(b) states is that the indirect utility generated by an incentive compatible direct mechanism admits two-sided directional derivatives in the direction δ_0^1 almost everywhere in the line segment $L(\theta_0, \theta_1)$. This property is used to infer an important relationship between the upper and lower subderivatives of the valuation function in the line segment connecting θ_0 to θ_1. To establish this relationship, define the functions $\overline{s}(\cdot)(\delta_0^1)$ and $\underline{s}(\cdot)(\delta_0^1)$ on $[0,1]$ by

$$\overline{s}(\alpha)(\delta_0^1) \;\equiv\; \overline{\mathrm{d}}v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha))(\delta_0^1), \qquad \underline{s}(\alpha)(\delta_0^1) \;\equiv\; \underline{\mathrm{d}}v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha))(\delta_0^1). \tag{6}$$

We make the following additional assumption to obtain a characterization of implementable allocation rules.

(M) Given an allocation rule $X\colon \Theta \to \mathcal{X}$, for every pair of types $\theta_0, \theta_1 \in \Theta$ the subderivative functions $\alpha \mapsto \overline{s}(\alpha)(\delta_0^1)$ and $\alpha \mapsto \underline{s}(\alpha)(\delta_0^1)$, defined on $[0,1]$ by expression (6), are $\mathcal{B}([0,1])$-measurable.

Our assumption (M) may be easily verified in some circumstances; this happens in Example 1, where for $\Theta = (0,1)$ the functions $\theta \mapsto \overline{\mathrm{d}}v(X(\theta),\theta)(1)$ and $\theta \mapsto \underline{\mathrm{d}}v(X(\theta),\theta)(1)$ are constant. At the end of this section, we discuss important cases where (M) is satisfied and relate our assumption to recent approaches in the literature.

Using (6), we define the *subderivative correspondence* $S(\cdot)(\delta_0^1)\colon [0,1] \rightrightarrows \mathbb{R}$ between types θ_0 and θ_1 by

$$S(\alpha)(\delta_0^1) \;=\; \big\{\, r \in \mathbb{R} \mid \overline{s}(\alpha)(\delta_0^1) \;\leq\; r \;\leq\; \underline{s}(\alpha)(\delta_0^1) \,\big\}. \tag{7}$$

The image $S(\alpha)(\delta_0^1)$ is empty for each α in $[0,1]$ for which $\overline{s}(\alpha)(\delta_0^1) > \underline{s}(\alpha)(\delta_0^1)$. Whenever the opposite inequality holds, $S(\alpha)(\delta_0^1)$ is a non-empty set containing all real numbers between the lower and the upper subderivatives of the valuation v with respect to types evaluated at $(X(\theta_0^1(\alpha), \theta_0^1(\alpha))$, where these subderivatives are taken in the direction δ_0^1. In such case, write the image of the subderivative correspondence as the closed interval $S(\alpha)(\delta_0^1) = [\overline{s}(\alpha)(\delta_0^1), \underline{s}(\alpha)(\delta_0^1)]$. Observe that since $\overline{\mathrm{d}}v(x,\theta)(\delta_0^1) = -\,\underline{\mathrm{d}}v(x,\theta)(-\,\delta_0^1)$, one has $S(\alpha)(-\,\delta_0^1) = -\,S(\alpha)(\delta_0^1)$.

The subderivative correspondence satisfies several important properties.

Lemma 3. *Assume (A1) to (A3) are satisfied, and suppose that (M) holds for the allocation rule $X\colon \Theta \to \mathcal{X}$. If X is implementable, then for every pair of distinct types $\theta_0, \theta_1 \in \Theta$, the subderivative correspondence $S(\cdot)(\delta_0^1)$ defined in* (7) *has non-empty images a.e. in* $[0,1]$*. Moreover, $S(\cdot)(\delta_0^1)$ is closed-valued, measurable and integrably bounded.*

[6] As an illustration, let $f(y_1, y_2) = |y_2|$, $\hat{y} = (0,0)$ and $y = (1,0)$. One sees that f is nowhere differentiable on the line segment connecting $\hat{y}$ to y. Krishna and Maenner (2001) present a more dramatic example.

Proof Suppose $p\colon \Theta \to \mathbb{R}$ is a payment scheme that implements X. Fix arbitrary types $\theta_0, \theta_1 \in \Theta, \theta_0 \neq \theta_1$. Then for every $\theta_0^1(\alpha) \in L(\theta_0, \theta_1)$, for any scalar r sufficiently small, the indirect utility function U generated by (X, p) satisfies

$$\begin{aligned} U(\theta_0^1(\alpha) + r\,\delta_0^1) \;-\; U(\theta_0^1(\alpha)) \;&\geq\; v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha) + r\,\delta_0^1) \;-\; p(\theta_0^1(\alpha)) \\ &\quad\;-\; v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha)) \;+\; p(\theta_0^1(\alpha)) \\ &=\; v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha) + r\,\delta_0^1) \;-\; v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha)). \end{aligned}$$

Thus, if $r > 0$ then it follows from the above expression that

$$\frac{v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha) + r\,\delta_0^1) - v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha))}{r} \;\leq\; \frac{U(\theta_0^1(\alpha) + r\,\delta_0^1) - U(\theta_0^1(\alpha))}{r}, \quad (8)$$

whereas if $r < 0$ we have

$$\frac{U(\theta_0^1(\alpha) + r\,\delta_0^1) - U(\theta_0^1(\alpha))}{r} \;\leq\; \frac{v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha) + r\,\delta_0^1) - v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha))}{r}. \quad (9)$$

By Lemma 2(b), U admits two-sided directional derivatives in the direction δ_0^1 almost everywhere in $L(\theta_0, \theta_1)$. Thus, taking the lower limit as $r \downarrow 0$ to (8) and the upper limit as $r \uparrow 0$ to (9), we infer that almost everywhere in $[0, 1]$ the following holds:

$$\overline{s}(\alpha)(\delta_0^1) \;\leq\; DU(\theta_0^1(\alpha))(\delta_0^1) \;\leq\; \underline{s}(\alpha)(\delta_0^1). \quad (10)$$

This shows that the subderivative correspondence $S(\cdot)(\delta_0^1)$ has non-empty images a.e. in $[0, 1]$, as desired.

Clearly, $S(\cdot)(\delta_0^1)$ is closed valued. To show that it is a measurable correspondence, define the correspondence $T\colon [0, 1] \rightrightarrows \mathbb{R}$ by $T(\alpha) = \{\overline{s}(\alpha)(\delta_0^1)\} \cup \{\underline{s}(\alpha)(\delta_0^1)\}$ when expression (10) is satisfied, and $T(\alpha) = \emptyset$, otherwise. Since the set $\{\, \alpha \in [0, 1] \mid T(\alpha) = \emptyset \,\}$ has zero measure, we deduce from our assumption (M) that T is a measurable correspondence, and therefore so is its convex hull $\operatorname{conv} T = S(\cdot)(\delta_0^1)$.[7] Further, notice that the type space is bounded and for every $x \in \mathcal{X}$, (A3) implies that $|\overline{\mathrm{d}}v(x, \theta_0^1(\alpha))(\delta_0^1)| \leq \ell \|\delta_0^1\|$ and similarly $|\underline{\mathrm{d}}v(x, \theta_0^1(\alpha))(\delta_0^1)| \leq \ell \|\delta_0^1\|$, for all $\alpha \in [0, 1]$. Hence, $\overline{s}(\cdot)(\delta_0^1)$ and $\underline{s}(\cdot)(\delta_0^1)$ are integrably bounded functions, which shows that $S(\cdot)(\delta_0^1)$ is an integrably bounded correspondence. ■

The subderivative correspondence $S(\cdot)(\delta_0^1)$ between θ_0 and θ_1 is said to be *regular* if it is non empty-valued a.e. in $[0, 1]$, closed-valued, measurable and integrably bounded. Thus, Lemma 3 states that if our assumption (M) is satisfied for an implementable allocation rule X, the subderivative correspondence between any two types generated by X is regular. In this case, the integral of $S(\cdot)(\delta_0^1)$ is a non-empty, closed interval. In

[7] Here we use two facts: (i) the (countable) union of measurable correspondences is measurable; (ii) the convex hull of a measurable correspondence is also measurable. See the references given in Section 2.2.

particular, $\alpha \mapsto \overline{s}(\alpha)(\delta_0^1)$ and $\alpha \mapsto \underline{s}(\alpha)(\delta_0^1)$ are integrable selections, with

$$\int_0^1 \overline{s}(\alpha)(\delta_0^1)\, d\alpha \;\leq\; \int_0^1 s(\alpha)(\delta_0^1)\, d\alpha \;\leq\; \int_0^1 \underline{s}(\alpha)(\delta_0^1)\, d\alpha$$

being satisfied for every integrable selection $\alpha \mapsto s(\alpha)(\delta_0^1)$ of $S(\cdot)(\delta_0^1)$.

Our main result is the following characterization theorem. It is understood that if $\theta_0 = \theta_1$, then $\overline{s}(\cdot)(\delta_0^1) = \underline{s}(\cdot)(\delta_0^1) = 0$.

Theorem 4. *Assume (A1) to (A3) are satisfied. Suppose the allocation rule $X\colon \Theta \to \mathcal{X}$ is such that (M) is also satisfied. The following statements are then equivalent.*

(a) The allocation rule $X\colon \Theta \to \mathcal{X}$ is implementable.

(b) For every subset $\{\theta_0, \theta_1, \theta_2\}$ of Θ, letting $\delta_n^m = \theta_m - \theta_n$ for $n, m = 0, 1, 2$, the subderivative correspondence $S(\cdot)(\delta_n^m)$ between θ_n and θ_m is regular and admits an integrable selection $\alpha \mapsto s^*(\alpha)(\delta_n^m)$ that satisfies the integral monotonicity condition:

$$v(X(\theta_m), \theta_m) - v(X(\theta_m), \theta_n) \;\geq\; \int_0^1 s^*(\alpha)(\delta_n^m)\, d\alpha \;\geq\; v(X(\theta_n), \theta_m) - v(X(\theta_n), \theta_n).$$

Moreover, these selections satisfy the path-integrability condition:

$$\int_0^1 s^*(\alpha)(\delta_0^1)\, d\alpha \;+\; \int_0^1 s^*(\alpha)(\delta_1^2)\, d\alpha \;+\; \int_0^1 s^*(\alpha)(\delta_2^0)\, d\alpha \;=\; 0.$$

Proof $(a) \Longrightarrow (b)$ Fix a subset $\{\theta_0, \theta_1, \theta_2\}$ of Θ. We first note that from Lemma 3, $S(\cdot)(\delta_n^m)$ is a regular correspondence. Denote $\theta_n^m(\alpha) = \theta_n + \alpha\, \delta_n^m$, for $\alpha \in [0, 1]$ and $n, m = 0, 1, 2$. Notice that from Lemma 2(b), the function μ_n^m defined on the interval $[0, 1]$ by $\mu_n^m(\alpha) = U(\theta_n^m(\alpha))$ is absolutely continuous, with $D\mu_n^m(\alpha) = DU(\theta_n^m(\alpha))(\delta_n^m)$ for almost all $\alpha \in [0, 1]$. Therefore, we have

$$\mu_n^m(1) - \mu_n^m(0) \;=\; U(\theta_m) - U(\theta_n) \;=\; \int_0^1 DU(\theta_n^m(\alpha))(\delta_n^m)\, d\alpha.$$

This expression is combined with (10) to obtain

$$\int_0^1 \overline{s}(\alpha)(\delta_n^m)\, d\alpha \;\leq\; U(\theta_m) \;-\; U(\theta_n) \;\leq\; \int_0^1 \underline{s}(\alpha)(\delta_n^m)\, d\alpha. \tag{11}$$

The convexity of the integral of the subderivative correspondence $S(\cdot)(\delta_n^m)$ provides us with the existence of an integrable selection $s^*(\cdot)(\delta_n^m)$ such that $\int_0^1 s^*(\alpha)(\delta_n^m)\, d\alpha =$

$U(\theta_m) - U(\theta_n)$. Moreover, from the proof of Lemma 2(a), we notice that

$$v(X(\theta_m), \theta_m) \; - \; v(X(\theta_m), \theta_n) \; \geq \; U(\theta_m) \; - \; U(\theta_n) \; = \; \int_0^1 s^*(\alpha)(\delta_n^m) \, d\alpha.$$

Exchanging the roles of θ_n and θ_m, one has

$$v(X(\theta_n), \theta_n) \; - \; v(X(\theta_n), \theta_m) \; \geq \; U(\theta_n) \; - \; U(\theta_m) \; = \; - \int_0^1 s^*(\alpha)(\delta_n^m) \, d\alpha.$$

The integral monotonicity condition is obtained combining these two equations.

Clearly, we have $(U(\theta_1) - U(\theta_0)) + (U(\theta_2) - U(\theta_1)) + (U(\theta_0) - U(\theta_2)) = 0$. Therefore, using the selection $s^*(\cdot)(\delta_n^m)$ from the subderivative correspondence $S(\cdot)(\delta_n^m)$ for each respective case, we immediately obtain the integrability condition.

(b) $\Longrightarrow$ *(a)* Fix a type $\theta_0 \in \Theta$. Define the payment rule $p \colon \Theta \to \mathbb{R}$ by

$$p(\theta_1) \; = \; v(X(\theta_1), \theta_1) \; - \; \int_0^1 s^*(\alpha)(\delta_0^1) \, d\alpha, \qquad \text{for all } \theta_1 \in \Theta \; .$$

Here $s^*(\cdot)(\delta_n^m)$ are integrable selections of the regular subderivative correspondences $S(\cdot)(\delta_n^m)$ for which the assumptions of the theorem are satisfied. We claim that X is implementable by p. Indeed, for any two types $\theta_1, \theta_2 \in \Theta$, the payment difference is

$$\begin{aligned} p(\theta_2) - p(\theta_1) \; &= \; v(X(\theta_2), \theta_2) - v(X(\theta_1), \theta_1) + \int_0^1 s^*(\alpha)(\delta_0^1) \, d\alpha + \int_0^1 s^*(\alpha)(\delta_2^0) \, d\alpha \\ &= \; v(X(\theta_2), \theta_2) \; - \; v(X(\theta_1), \theta_1) \; - \; \int_0^1 s^*(\alpha)(\delta_1^2) \, d\alpha, \end{aligned}$$

where the first equality follows from the fact that $s^*(\cdot)(\delta_2^0) = - \, s^*(\cdot)(\delta_0^2)$, and the last equality follows from the integrability condition. Using this expression, we deduce from the integral monotonicity condition that

$$\begin{aligned} &\big\{ v(X(\theta_1), \theta_1) \; - \; p(\theta_1) \big\} \; - \; \big\{ v(X(\theta_2), \theta_1) \; - \; p(\theta_2) \big\} \\ &\quad = \; v(X(\theta_1), \theta_1) \; - \; v(X(\theta_2), \theta_1) \; + \; p(\theta_2) \; - \; p(\theta_1) \\ &\quad = \; v(X(\theta_2), \theta_2) \; - \; v(X(\theta_2), \theta_1) \; - \; \int_0^1 s^*(\alpha)(\delta_1^2) \, d\alpha \; \geq \; 0. \end{aligned}$$

Hence, it follows that $v(X(\theta_1), \theta_1) - p(\theta_1) \geq v(X(\theta_2), \theta_1) - p(\theta_2)$. Since θ_1 and θ_2 were arbitrarily chosen, this shows that the payment rule p implements X, as desired. ∎

It is not difficult to see that one can replace the global conditions of part (b) of Theorem 4 with their local versions, an approach that was introduced by Archer and Kleinberg (2008) for the case of valuation functions that are linear in types (see Berger, Müller, and Naeemi (2009) for the case of valuations that are convex in the type space).

This relies on the fact that for all types $\theta_0, \theta_1, \theta_2$ in Θ, the line segments $L(\theta_n, \theta_m)$ $(n, m = 0, 1, 2)$ and their convex hull are compact subsets of $\mathbb{R}^k$. Therefore, one can replace an open cover of any such set with a finite subcover to obtain the desired conditions. We state this formally in the next proposition, whose proof can be adapted from the arguments of Lemmas 3.5 and 3.2 in Archer and Kleinberg (2008).

Proposition 5. *Assume (A1) to (A3) are satisfied. Assume in addition that (M) is satisfied for the allocation rule $X\colon \Theta \to \mathcal{X}$. The following are equivalent:*

(a) The allocation rule $X\colon \Theta \to \mathcal{X}$ is implementable.

(b) For each $\theta_0 \in \Theta$, there exists an open neighborhood O of θ_0 such that for every $\theta_1, \theta_2 \in O$, for $n, m = 0, 1, 2$, the subderivative correspondence $S(\cdot)(\delta_n^m)$ is regular and admits an integrable selection $s^*(\cdot)(\delta_n^m)$ satisfying the local integral monotonicity condition:

$$v(X(\theta_m), \theta_m) - v(X(\theta_m), \theta_n) \;\geq\; \int_0^1 s^*(\alpha)(\delta_n^m)\, d\alpha \;\geq\; v(X(\theta_n), \theta_m) - v(X(\theta_n), \theta_n).$$

Further, the selections $s^*(\cdot)(\delta_n^m)$ satisfy the local integrability condition:

$$\int_0^1 s^*(\alpha)(\delta_0^1)\, d\alpha \;+\; \int_0^1 s^*(\alpha)(\delta_1^2)\, d\alpha \;+\; \int_0^1 s^*(\alpha)(\delta_2^0)\, d\alpha \;=\; 0.$$

We mention that the integrability condition is trivially satisfied in one-dimensional type spaces, in which case the characterization of implementable allocation rules is achieved through the integral monotonicity condition alone, provided (A1) to (A3) and (M) are in place.

Example 1 (continued). Write the regular subderivative correspondence generated by the efficient allocation rule X^* as $\theta \rightrightarrows S(\theta) = [-1, 1]$. Fix a type $\theta_0 \in (0, 1)$. Since $X^*(\theta) = \theta$, the integral monotonicity condition can be expressed as

$$|\theta_1 - \theta_0| \;\geq\; \int_{\theta_0}^{\theta_1} s(\theta)\, d\theta \;\geq\; -|\theta_1 - \theta_0|, \qquad \text{all } \theta_1 \in (0, 1).$$

Moreover, since $\int_{\theta_0}^{\theta_1} \underline{s}(\theta)\, d\theta = |\theta_1 - \theta_0|$ and $\int_{\theta_0}^{\theta_1} \overline{s}(\theta)\, d\theta = -|\theta_1 - \theta_0|$, it follows that X^* can be implemented by *any* payment scheme of the form

$$p(\theta_1) \;=\; -\int_{\theta_0}^{\theta_1} s(\theta)\, d\theta,$$

where $\theta \mapsto s(\theta)$ is an integrable selection of the subderivative correspondence $\theta \rightrightarrows S(\theta)$; three such schemes were given in the Introduction. Observe that the indirect utility function generated by an incentive compatible mechanism may be (strictly) increasing, (strictly) decreasing, or constant in types. Notice that equilibrium payoffs, hence equilibrium revenues, are not equivalent up to an additive constant. □

When is (M) satisfied? In some circumstances, it is immediate to verify this condition directly (see Example 1 and also Example 11 in Section 4). The next proposition may serve as a useful tool for this task, when the allocation set is a metric space.

Proposition 6. *Assume that (A1) to (A3) are satisfied. Assume, in addition, that the allocation set $\mathcal{X}$ is a complete separable metric space. Suppose that for every direction $\delta \in \mathbb{R}^k$, the following conditions hold:*

(a) For every $x \in \mathcal{X}$, the functions $\overline{\mathrm{d}}v(x,\cdot)(\delta)$ and $\underline{\mathrm{d}}v(x,\cdot)(\delta)$ are $\mathcal{B}(\Theta)$-measurable.

(b) For every $\theta \in \Theta$, the functions $\overline{\mathrm{d}}v(\cdot,\theta)(\delta)$ and $\underline{\mathrm{d}}v(\cdot,\theta)(\delta)$ are continuous on $\mathcal{X}$.

Then for any measurable allocation rule $X \colon \Theta \to \mathcal{X}$, (M) is satisfied.

Proof Fix a direction $\delta \in \mathbb{R}^k$ for the remainder of the proof. It suffices to show that $\theta \mapsto \overline{\mathrm{d}}v(X(\theta),\theta)(\delta)$ and $\theta \mapsto \underline{\mathrm{d}}v(X(\theta),\theta)(\delta)$ are $\mathcal{B}(\Theta)$-measurable. Since the allocation rule X is measurable, there exists a sequence of simple measurable functions $\{X_n \colon \Theta \to \mathcal{X}\}_{n=1}^{\infty}$ that converges pointwise to X. By assumption, for allocation every allocation x, the function $\theta \mapsto \overline{\mathrm{d}}v(x,\theta)(\delta)$ is measurable, hence it follows that for each positive integer n the function $\theta \mapsto \overline{\mathrm{d}}v(X_n(\theta),\theta)(\delta)$ is also measurable. We now use the continuity of $\overline{\mathrm{d}}v(\cdot,\theta)(\delta)$ on $\mathcal{X}$ to obtain that, for each $\theta \in \Theta$, $\overline{\mathrm{d}}v(X(\theta),\theta)(\delta) = \lim_{n\to\infty} \overline{\mathrm{d}}v(X_n(\theta),\theta)(\delta)$. This shows that the function $\theta \mapsto \overline{\mathrm{d}}v(X(\theta),\theta)(\delta)$ is the pointwise limit of $\mathcal{B}(\Theta)$-measurable functions, which give us the desired conclusion. The argument for the lower subderivative function is analogous. ■

In some environments of special economic interest, (M) will follow directly from the structure of the design problem, as we now discuss.

Consider, as in Archer and Kleinberg (2008), a setting with a bounded allocation set $\mathcal{X} \subseteq \mathbb{R}^k$ and $\boldsymbol{\mathcal{X}} = \mathcal{B}(\mathcal{X})$, with an open, bounded convex type space $\Theta \subseteq \mathbb{R}^k$, and with valuation functions that are linear in types, so that $\theta \mapsto v(x,\theta) = x \cdot \theta$ for each allocation x. Notice that (A1) to (A3) are satisfied. Let $X \colon \Theta \to \mathcal{X}$ be any measurable, integrably bounded allocation rule. One readily sees that the derivative of v with respect to θ evaluated at $(X(\theta),\theta)$ is $Dv(X(\theta),\theta) = X(\theta)$, therefore (M) is satisfied and our characterization theorem applies.

Berger, Müller, and Naeemi (2009) deal with an arbitrary allocation set, a convex type space in $\mathbb{R}^k$, and valuations that are convex functions of types. In this case, the assumptions of Proposition 6 may be violated, as the directional derivative of a convex function may be discontinuous. However, directional derivatives of (one-dimensional) convex functions are sufficiently well-behaved for our characterization result to be applicable. First, a preliminary result taken from Hildenbrand (1974, (7) p. 42).

Proposition 7. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}})$ be a measurable space and $f\colon \mathcal{X} \times [0,1]$ be a bounded real-valued function. Suppose the following conditions hold:*

(a) For every $\alpha \in [0,1]$, the function $f(\cdot, \alpha)$ is $\boldsymbol{\mathcal{X}}$-measurable.

(b) For every $x \in \mathcal{X}$, the function $f(x, \cdot)$ is right-continuous.

Then the function f is $\boldsymbol{\mathcal{X}} \otimes \mathcal{B}([0,1])$-measurable.

In the above proposition, $\boldsymbol{\mathcal{X}} \otimes \mathcal{B}([0,1])$ denotes the product σ-algebra of $\boldsymbol{\mathcal{X}}$ and $\mathcal{B}([0,1])$. We shall use this to prove the following result, which under a mild measurability requirement on the upper and lower subderivative functions, allows us to apply Theorem 4 to the case of valuations that are convex in types.

Proposition 8. *Assume that (A1) to (A3) are satisfied. Suppose also that for every allocation $x \in \mathcal{X}$, the function $\theta \mapsto v(x,\theta)$ is convex in Θ, and that for every direction $\delta \in \mathbb{R}^k$ and every type $\theta \in \Theta$, both $\overline{\mathrm{d}}(\cdot,\theta)(\delta)$ and $\underline{\mathrm{d}}(\cdot,\theta)(\delta)$ are $\boldsymbol{\mathcal{X}}$-measurable. Then for any measurable allocation rule $X\colon \Theta \to \mathcal{X}$, assumption (M) is satisfied.*

Proof Fix any $\theta_0, \theta_1 \in \Theta$. From (A3) and the convexity of the valuation with respect to types, it follows that for every $x \in \mathcal{X}$, $\overline{\mathrm{d}}v(x, \theta_0^1(\alpha))(\delta_0^1)$ is equal to the right-derivative of the convex scalar function $\alpha \mapsto w(\alpha; x, \delta_0^1) = v(x, \theta_0^1 + \alpha\, \delta_0^1)$ for every $\alpha \in [0,1)$. It follows that the function $\alpha \mapsto \overline{\mathrm{d}}v(x,\theta_0^1(\alpha))(\delta_0^1)$ is right-continuous on $[0,1)$. Since by assumption $\overline{\mathrm{d}}v(\cdot,\theta_0^1(\alpha))(\delta_0^1)$ is $\boldsymbol{\mathcal{X}}$-measurable for every $\alpha \in [0,1]$, we use Proposition 7 to infer that the function $\overline{\mathrm{d}}(\cdot,\cdot)(\delta_0^1)$ is $\boldsymbol{\mathcal{X}} \otimes \mathcal{B}([0,1])$-measurable. Since the allocation rule X is measurable, it is seen that the function $\alpha \mapsto (X(\theta_0^1(\alpha)), \theta_0^1(\alpha))$ is $\mathcal{B}([0,1])$-measurable, from which we obtain the measurability of $\alpha \mapsto \overline{s}(\alpha)(\delta_0^1) = \overline{\mathrm{d}}v(X(\theta_0^1(\alpha)), \theta_0^1(\alpha))(\delta_0^1)$ by noticing that the composition of measurable functions is also measurable. The argument for $\underline{s}(\cdot)(\delta_0^1)$ is similar (except that now we use the left-derivative of the respective convex scalar function. ■

4 A generalized Revenue Equivalence theorem

Our characterization result in Section 3 does *not* imply that the indirect utility generated by an incentive compatible mechanism is completely determined by the allocation rule alone. Thus the standard version of the Revenue Equivalence theorem may fail. On the other hand, if X is implemented by a payment rule p, then from Theorem 4 one infers that the difference in the indirect utility generated by (X,p) at θ_1 and θ_0 is bounded by X alone, since the upper and lower bounds of $U(\theta_1) - U(\theta_0)$ are independent of p. We use this fact to obtain a generalized version of the Revenue Equivalence Principle.

Proposition 9. *Assume (A1) to (A3) hold and let $X\colon \Theta \to \mathcal{X}$ be an allocation rule for which (M) is satisfied. Consider two payment rules, $p\colon \Theta \to \mathbb{R}$ and $p'\colon \Theta \to \mathbb{R}$, that* *implement X. Let U and U′ denote the indirect utility functions generated by the direct mechanisms (X, p) and (X, p'), respectively. For all $\theta_0, \theta_1 \in \Theta$, one has*

$$\left|\big(U(\theta_1) - U(\theta_0)\big) \;-\; \big(U'(\theta_1) - U'(\theta_0)\big)\right| \;\leq\; \int_0^1 \left\{\underline{s}(\alpha)(\delta_0^1) \;-\; \overline{s}(\alpha)(\delta_0^1)\right\} d\alpha. \qquad (12)$$

Proof By Theorem 4, since X is implemented by p and by p', it follows that both indirect utility functions U and U' must satisfy expression (11), from where we deduce the inequalities

$$\int_0^1 \overline{s}(\alpha)(\delta_0^1)\, d\alpha \;\leq\; U(\theta_1) - U(\theta_0) \;\leq\; \int_0^1 \underline{s}(\alpha)(\delta_0^1)\, d\alpha, \qquad \text{and}$$

$$-\int_0^1 \underline{s}(\alpha)(\delta_0^1)\, d\alpha \;\leq\; U'(\theta_0) - U'(\theta_1) \;\leq\; -\int_0^1 \overline{s}(\alpha)(\delta_0^1)\, d\alpha.$$

Equation (12) now follows by adding up these two expressions. ■

Let X be implementable, and fix a type $\theta_0 \in \Theta$. Then, from the proof of Theorem 4, for every type θ_1 one can use an integrable selection from the regular subderivative correspondence $S(\cdot)(\delta_0^1)$, satisfying the integral monotonicity and the integrability conditions, to define a payment scheme that implements X. Suppose $s(\cdot)(\delta_0^1)$ and $s'(\cdot)(\delta_0^1)$ are two such selections. Thus, the payment schemes p and p' defined by $p(\theta_1) = v(X(\theta_1), \theta_1) - \int_0^1 s(\alpha)(\delta_0^1)\, d\alpha$ and $p'(\theta_1) = v(X(\theta_1), \theta_1) - \int_0^1 s'(\alpha)(\delta_0^1)\, d\alpha$, all $\theta_1 \in \Theta$, implement X and generate indirect utilities for which $U(\theta_0) = U'(\theta_0) = 0$. In this case, (12) reduces to

$$|U(\theta_1) - U'(\theta_1)| \;\leq\; \int_0^1 \left\{\underline{s}(\alpha)(\delta_0^1) \;-\; \overline{s}(\alpha)(\delta_0^1)\right\} d\alpha.$$

This last implies that, given two incentive compatible mechanisms for which the indirect utility levels of type θ_0 are zero, the difference in the indirect utility functions is bounded by the allocation rule alone (since X alone determines the upper and lower subderivative functions), even though this difference in equilibrium payoff does not vanish.

Notice also that from the convexity of the integral of the subderivative correspondence, given $\lambda \in [0, 1]$, for every $\theta_1 \in \Theta$ there exists an integrable selection $s''(\cdot)(\delta_0^1)$ such that $\int_0^1 s''(\alpha)(\delta_0^1) = \lambda \int_0^1 s(\alpha)(\delta_0^1) + (1 - \lambda) \int_0^1 s'(\alpha)(\delta_0^1)$. Clearly, $s''(\cdot)(\delta_0^m)$ satisfies the integral monotonicity and the integrability conditions. Thus, the payment scheme $p''\colon \Theta \to \mathbb{R}$ defined by

$$p''(\theta_1) \;=\; v(X(\theta_1), \theta_1) \;-\; \int_0^1 s''(\alpha)(\delta_0^1)\, d\alpha \;=\; \lambda p(\theta_1) \;+\; (1 - \lambda) p'(\theta_1),$$

all $\theta_1 \in \Theta$, implements X as well. The preceding argument shows the following.

Corollary 10. *If U and U' are indirect utility functions generated by two direct mechanisms sharing the same allocation rule X and assigning $U(\theta_0) = U'(\theta_0) = 0$ for some $\theta_0 \in \Theta$, then for every $\lambda \in [0,1]$ there exists a direct mechanism (X, p'') such that $U'' = \lambda U + (1-\lambda)U'$.*

The following example illustrates our generalized version of Revenue Equivalence.

Example 11. The allocation set is $\mathcal{X} = (0,1)$ and the type space is $\Theta = (0,1)$. The agent's valuation function $v\colon \mathcal{X} \to \Theta$ is given by

$$v(x,\theta) \;=\; \begin{cases} \theta x, & \text{if } x \le \theta; \\ 2\theta^2 \;-\; \theta x, & \text{if } x > \theta. \end{cases}$$

One can think of $x \in \mathcal{X}$ as the quantity traded of some good; our agent has positive marginal utility θ for the first θ units, and negative marginal utility $-\theta$ for additional amounts. Notice that for each alternative x, the function $\theta \mapsto v(x,\theta)$ is neither convex on Θ nor differentiable at $\theta = x$. It is however Lipschitz continuous, with Lipschitz constant $\ell(x) = 3x$. Thus (A1) to (A3) are satisfied.

The efficient allocation rule X^* selects $X^*(\theta) = \theta$, for all $\theta \in \Theta$. Observe that at every equilibrium point $(X^*(\theta), \theta)$, one has $\overline{\mathrm{d}}v(X^*(\theta),\theta) = \theta$ and $\underline{\mathrm{d}}v(X^*(\theta),\theta) = 3\theta$, hence (M) is also in place. Define the subderivative correspondence $S\colon \Theta \rightrightarrows \mathbb{R}$ by

$$S(\theta) \;=\; \big[\,\overline{\mathrm{d}}v(X^*(\theta),\theta), \underline{\mathrm{d}}v(X^*(\theta),\theta)\,\big] \;=\; \big[\theta, 3\theta\big].$$

Fix an arbitrary type $\theta_0 \in (0,1)$ for the remainder of the example. Consider the selection $\theta \mapsto \overline{s}(\theta) = \theta$. It can be verified that for every $\theta_1 \in (0,1)$, the following holds:

$$v(X^*(\theta_1),\theta_1) - v(X^*(\theta_1),\theta_0) \;\ge\; \int_{\theta_0}^{\theta_1} \overline{s}(\theta)\, d\theta \;\ge\; v(X^*(\theta_0),\theta_1) - v(X^*(\theta_0),\theta_0).$$

Thus, the integral monotonicity condition is satisfied. We infer from Theorem 4 that the payment rule $\overline{p}$, defined on Θ by $\overline{p}(\theta) = \frac{1}{2}(\theta^2 + \theta_0^2)$, all $\theta \in \Theta$, implements X^*. Indeed, suppose our agent is of type θ_1; reporting θ_2 yields her to payoffs of the form:

$$v(X^*(\theta_2),\theta_1) \;-\; \overline{p}(\theta_2) \;=\; \begin{cases} \theta_1\theta_2 \;-\; \frac{1}{2}\theta_2^2 \;-\; \frac{1}{2}\theta_0^2, & \text{if } \theta_2 \le \theta_1; \\ 2\theta_1^2 \;-\; \theta_1\theta_2 \;-\; \frac{1}{2}\theta_2^2 \;-\; \frac{1}{2}\theta_0^2, & \text{if } \theta_2 > \theta_1. \end{cases}$$

The above expression is strictly increasing in θ_2 for $\theta_2 < \theta_1$, strictly decreasing in θ_2 for $\theta_2 > \theta_1$, and reaches its maximum at $\theta_2 = \theta_1$. Thus, truth-telling is an equilibrium strategy. Notice that the agent's indirect utility function $\overline{U}$ generated by $(X^*, \overline{p})$ satisfies $\overline{U}(\theta_1) = \frac{1}{2}(\theta_1^2 - \theta_0^2)$.

Consider instead the selection $\theta \mapsto s^*(\theta) = 2\theta$. The reader can verified that the integral monotonicity condition is satisfied, and that the constant payment rule p^*, defined on Θ by $p^*(\theta) = \theta_0^2$, implements X^* and generates an indirect utility U^* for which

$U^*(\theta_1) = \theta_1^2 - \theta_0^2$. Notice that the difference between indirect utilities $\overline{U}$ and U^* is bounded by the difference between the integral of the selections $\theta \mapsto \underline{s}(\theta) = 3\theta$ and $\theta \mapsto \overline{s}(\theta) = \theta$, since

$$\left|\tfrac{1}{2}(\theta_1^2 - \theta_0^2)\right| \;=\; \left|\overline{U}(\theta_1) - U^*(\theta_1)\right| \;\leq\; \int_{\theta_0}^{\theta_1} \left\{\underline{s}(\theta) - \overline{s}(\theta)\right\} d\theta \;=\; |\theta_1^2 - \theta_0^2|.$$

We point out that in this example, the integral monotonicity condition is *not* satisfied for every integrable selection of the correspondence $S(\theta) = [\theta, 3\theta]$. In particular, for $\theta \mapsto \underline{s}(\theta) = 3\theta$, monotonicity is violated. It follows that the payment scheme $\underline{p}$ associated with $\underline{s}$, where $\underline{p}(\theta) = \frac{3}{2}\theta_0^2 - \frac{1}{2}\theta^2$ for all $\theta \in \Theta$, does not implement X^*. Indeed, for any $\theta_1, \theta_2 \in \Theta$, the agent payoffs of reporting θ_2 when her true type is θ_1 are

$$v(X^*(\theta_2), \theta_1) \;-\; \underline{p}(\theta_2) \;=\; \begin{cases} \theta_1\theta_2 \;-\; \frac{3}{2}\theta_0^2 \;+\; \frac{1}{2}\theta_2^2, & \text{if } \theta_2 \leq \theta_1; \\ 2\theta_1^2 \;-\; \theta_1\theta_2 \;-\; \frac{3}{2}\theta_0^2 \;+\; \frac{1}{2}\theta_2^2, & \text{if } \theta_2 > \theta_1. \end{cases}$$

These payoffs are increasing in θ_2 everywhere on Θ, thus our agent has incentives to overstate her type. □

It is clear that in Example 11, as in Example 1, the standard version of Revenue Equivalence fails, since the different payment schemes that can be used to implement X^* are not equivalence up to a constant, and thus the indirect utility of our agent depend on the payment scheme as well as on the allocation rule X^*. Nonetheless, if (A1) to (A3) and (M) are satisfied, one can use Proposition 9 to establish a precise range of values such that any incentive compatible mechanism generates indirect utility functions with differences lying inside this range. Clearly, if for all types θ_0, θ_1, the subderivative correspondence $S(\cdot)(\delta_0^1)$ between θ_0 and θ_1 is single-valued a.e. on $[0,1]$, then the standard version of Revenue Equivalence is obtained.

Proposition 12. *Assume that (A1) to (A3) are satisfied. Suppose (M) holds for the implementable allocation rule $X\colon \Theta \to \mathcal{X}$, and let $p\colon \Theta \to \mathbb{R}$ and $p'\colon \Theta \to \mathbb{R}$ be two payment rules that implement X, with U and U' denoting the indirect utility functions generated by (X,p) and (X,p'), respectively. If for every pair $\theta_0, \theta_1 \in \Theta$, one has $\underline{s}(\alpha)(\delta_0^1) = \overline{s}(\alpha)(\delta_0^1)$ for almost all $\alpha \in [0,1]$, then U and U' differ at most by a constant.*

Proof Suppose for all θ_0, θ_1 in Θ, it is the case that $\underline{s}(\alpha)(\delta_0^1) = \overline{s}(\alpha)(\delta_0^1)$ a.e. in $[0,1]$. Readily from expression (12), it follows that $U(\theta_1) - U(\theta_0) = U'(\theta_1) - U'(\theta_0)$. Therefore, the indirect utilities U and U' differ at most by a constant. ■

If the valuation function $\theta \mapsto v(x,\theta)$ is fully differentiable in Θ, for every $x \in \mathcal{X}$, then the standard version of Revenue Equivalence follows from Proposition 12. Revenue Equivalence also holds when the function $\theta \mapsto v(x,\theta)$ is convex in Θ, for all allocations $x \in \mathcal{X}$, and admits bounded directional derivatives everywhere in the type space (in which case the subderivatives are equal to the directional derivatives), since one has that for all $\hat{\theta} \in \Theta$, all $\delta \in \mathbb{R}^k$,

$$\underline{\mathrm{d}}v(x,\hat{\theta})(\delta) = \lim_{r\uparrow 0} \frac{v(x,\hat{\theta}+r\delta) - v(x,\hat{\theta})}{r} \leq \lim_{r\downarrow 0} \frac{v(x,\hat{\theta}+r\delta) - v(x,\hat{\theta})}{r} = \overline{\mathrm{d}}v(x,\hat{\theta})(\delta).$$

Recalling that equation (10) requires that at almost all equilibrium points $(X(\theta), \theta)$ the reverse inequality holds, one sees that the condition of Proposition 12 is in place.

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