THE SIMPLE BEHAVIOR OF LARGE MECHANISMS

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Abstract. In this paper we compare the equilibria of a mechanism with a large finite number of participants to the equilibria of an analogous mechanism featuring a nonatomic continuum of participants. We show that the equilibrium strategies of the two models will converge as the number of participants in the large finite mechanism goes to infinity under mild technical conditions. Given that these conditions hold, we can use tractable nonatomic models to analyze the large market behavior of otherwise intractable game-theoretic models. We apply these results to show that the equilibrium of a uniform price auction with a large number of agents and goods can be approximated by a nonatomic exchange economy. From this approximation, we are able to show that the uniform price auction is approximately efficient with a large number of participants even when agents have complementary preferences for multiple units or preferences have a common value component, both cases that have resisted analysis using game-theoretic techniques. In a second application, we show that the Markov perfect equilibria of a dynamic market competition model approaches the dynamic competitive equilibria of a game with a continuum of agents in the limit as the number of competitors in the large finite model approaches infinity.

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1. Introduction

Market mechanisms are analyzed using one of two paradigmatic classes of models. In game-theoretic models, finite sets of atomic agents choose optimal strategies given their beliefs about the actions of the other actors. Following Aumann [7], another framework for modeling economies with a large number of agents is the equilibria of nonatomic models with a continuum of agents. In nonatomic models, the agents take actions that are optimal given economic aggregates, such as prices or firm production, that are consistent with the agents' actions but exogenous to any single agent's decision. A researcher's choice to use a game-theoretic or a nonatomic model is driven by whether strategic interactions between the agents are important factors in determining the economic outcomes.

Intuition suggests that if the economic outcomes are continuous in the aggregate distribution of agent actions,[1] then a single agent will have negligible influence on outcomes as the number of agents increases. This continuity implies that the game-theoretic equilibria may converge to the equilibria of the corresponding nonatomic game as the number of agents increases. As a consequence of this convergence, the decision about whether to use a game-theoretic or a nonatomic model will have an inconsequential effect on the predicted economic outcomes if the market is sufficiently large. However, it has not been obvious under what conditions this intuition is valid in the context of complex market models such as double auctions. Therefore, researchers often elect to use game-theoretic models for large markets and show through an analysis of the equilibria that economic aggregates become exogenous to a single agent's decisions in the limit as the number of agents approaches infinity. Due to the difficulty of finding the game-theoretic equilibria, researchers are forced to place restrictions on the model that are not dictated by the economic primitives.

A primary contribution of this paper is to outline formal conditions under which nonatomic models yield economic predictions that approximate those of analogous game-theoretic models with a large finite set of agents. The two principal requirements are that the model satisfy a semi-anonymity and a continuity condition. The semi-anonymity condition requires that an agent's outcome in the market depends only on a message he sends the market authority (i.e. a demand curve) and the market aggregates (i.e. market clearing prices) that are determined by the distribution of announcements of the agents in the market. The continuity condition demands that an agent's market outcome be continuous in the distribution of announcements of the agents in the economy. As the distribution

[1] Given a set of agent actions $\{x_1, ..., x_N\}$, the realized aggregate distribution of agent actions is an empirical cumulative distribution function defined as $F(x) = \sum_{i=1}^{N} 1\{x_i \leq x\}$ where $1\{x_i \leq x\}$ is an indicator function for the event $\{x_i \leq x\}$.

of agent messages is an element of an infinite dimensional space, an important step in the development of our conditions is identifying the form of continuity required. Given that the semi-anonymity and continuity conditions can both be verified from the economic primitives of the market model prior to discovering an equilibrium of the economy, there is no need for the researcher to engage in the arduous task of game-theoretic analysis to establish asymptotic equivalence of the game-theoretic and nonatomic equilibria.

This paper describes the relationship between large finite economies and their nonatomic analogs with three propositions. First, for any positive epsilon, we show that equilibria of the nonatomic model are epsilon Bayesian-Nash equilibria of sufficiently large finite mechanisms. Second, for any positive epsilon, we show that the exact Bayesian-Nash equilibria of a sufficiently large finite economy are epsilon equilibria of the analogous nonatomic economy. Finally, we show that the set of equilibrium strategies is upper hemicontinuous in the number of agents participating in the mechanism. We provide an example that shows that the equilibrium strategies are not lower hemicontinuous in this limit. Therefore, for sufficiently large finite mechanisms, the equilibrium predictions of the large finite economy are approximately equal to the predictions of the nonatomic model in both the payoff and strategy spaces. Throughout this work we treat the nonatomic models as stylized simplifications of more realistic large finite market models. We focus on applications where the nonatomic approximation allows us to study previously intractable problems.

The first novel application we develop is an analysis of large uniform price auction markets in which bidders may have complementary preferences for multiple homogenous goods. Uniform price auctions are one of the most common allocation mechanisms in practice with applications ranging from the sale of US Treasury Bills to BHP Billton diamond auctions. The game-theoretic equilibria of uniform price auctions predict that inefficiencies can occur as a result of agents withholding demand in order to reduce the market clearing price (see, for example, Ausubel and Crampton [11]). The equilibria of large uniform price auctions have been studied only in the case where agents have declining marginal valuations for successive units and homogenous goods are auctioned (Swinkels [87]), but the behavior of large uniform price auctions wherein agents view successive units as complements remains intractable when approached with game-theoretic techniques. However, the nonatomic form of the market is a general equilibrium exchange economy, which admits easily characterized equilibria even under nonconvex preferences. Given that equilibria of these general equilibrium models are efficient and generically possess a unique market clearing price, we can use our approximation results to show that the equilibria of the large finite uniform price auction admit approximately truthful strategies and have vanishing inefficiencies in the limit as the number of agents increases to infinity even in the case where agents treat successive units as complementary goods or the agents view different types of goods for auction as complements.

Although the focus of this work is the theoretical analysis of static markets, our techniques can be modified to allow us to analyze large stochastic games. Ericson and Pakes' [29] model of strategic interaction within industries with entry and exit is widely utilized by industrial organization economists for structural estimation. However, a significant difficulty of employing this model for empirical work is that, even under the restriction to Markov perfect equilibria, the estimation becomes intractable with more than a handful of firms. We employ our basic intuition regarding continuity and semi-anonymity to show that dynamic competitive equilibria of the nonatomic form of the dynamic game approximate Markov perfect equilibria of the large finite dynamic game. Dynamic competitive equilibria are often tractable to estimate for economies with a large number of agents. A particular interest are stationary equilibria, which are dynamic competitive equilibria with an unchanging state of the economy. Since an agent's strategy in a stationary equilibrium is a function only of his type and not of the aggregate distribution of types in the model, the estimation task escapes the curse of dimensionality even in very large economies. Our analysis does not rely on the details of the Ericson et al. model and can find application in any dynamic game model that admits stationary equilibria.

The application to dynamic games is of additional interest to economists who are concerned that the dynamic competitive equilibria computed for a nonatomic macroeconomic model may not faithfully represent the behavior of a large finite form of the model. By studying the equilibria of nonatomic models, macroeconomists have in effect been assuming that the nonatomic equilibria are adequate approximations of game-theoretic equilibria of large finite models. Our framework provides macroeconomic theorists with conditions under which this assumption is valid, and these conditions can be verified through inspection of the model primitives.

1.1. **Previous Literature.** This is not the first work to outline general conditions under which game-theoretic equilibria approach nonatomic equilibria as the number of agents increases. However, the prior literature on the relationship between large finite and nonatomic games does not emphasize the use of nonatomic games as a framework for tractably analyzing complex game-theoretic models. One of the barriers to the employment of this earlier work is that it is not formulated in a fashion that makes it amenable for application to general models. Our focus on a static mechanism design framework allows us to formulate and test our conditions in a wide array of applied settings.

Green [35], and later Sabourian [81], provide sufficient conditions for the set of nonatomic subgame perfect equilibria (SPE) and the SPE of the analogous large finite dynamic game to be approximately the same. The Green and Sabourian proof techniques require stringent continuity assumptions that have hampered the application of their analyses, while our proof technique requires weaker continuity assumptions that arise naturally in most mechanism design settings. In addition, some of the limit results in these papers require the convergence of the large finite games as an assumption, whereas convergence is a consequence of the economic primitives of our model.

A number of papers have sought to characterize equilibrium existence results for classes of nonatomic games, often attaining as a corollary that the exact equilibria of the nonatomic game are approximate equilibria of a large finite version of the game.[2] Much of this work employs nonstandard extensions to the familiar mathematical structures of real analysis.[3] One of the benefits of the framework we adopt below is that, while it is sufficiently general to capture many of the market and mechanism models of interest, the proofs are based on familiar topological notions of continuity.

Al-Najjar ([1], [2]) has studied the relationship between large finite economies and their nonatomic equivalents in a framework that is similar to the model discussed in this work. The principal goal of these works is to clarify difficult technical issues that have plagued the literature on generic nonatomic games. Unfortunately, the primitives assumed in these analyses are restricted in such a way that the results cannot be straightforwardly generalized to many of the models of interest to market designers and applied economists.

Al-Najjar et al. [3] prove that the fraction of players who are $\alpha-$pivotal has a bound of the form $O\left(\frac{1}{\sqrt{N}}\right)$.[4] Intuitively, for small α, showing an agent is not $\alpha-$pivotal is equivalent to showing that she has a negligible effect on the mechanism outcome. Al-Najjar et al. provide a limit on the fraction of $\alpha-$pivotal players, but allow a potentially large number of players to retain significant strategic choices for large N. The results of Al-Najjar et al. do not require a continuity condition on the mechanism, which is the force that drives the uniform limit on the pivotality of the players in our framework. The sharpest results

[2] Caramona [26] show the equivalence of many of these results and obtains a corollary of this form.

[3] A summary of results employing the techniques of nonstandard analysis is contained in Khan and Sun [44].

[4] This result is similar in spirit to Postlewaite and Schmeidler [72] who study a specific market game meant to capture the price formation process underlying general equilibrium exchange economies. Postlewaite and Schmeidler show in their setting that for any $\varepsilon > 0$ there exists N^* such that if $N > N^*$ agents play the market game, in equilibrium at most a fraction ε of the agents do not play ε-Walrasian responses. In this context, Walrasian responses are pricetaking responses as played in a general equilibrium exchange economy. This, of course, does *not* imply approximate efficiency as the fraction ε of nonoptimizers could control most of the resources and be injured badly in the approximate eqm.

in Al-Najjar et al. rely on the finiteness of the agents' action space. While a form of their results applies with infinite action spaces given a modification of the strategy space, the severity of the restrictions employed to generate this result is not clear within the context of mechanisms admitting infinite action spaces.

Kalai [43] provides a study of semi-anonymous large games in order to show the equilibria of games with finite action and type spaces are robust to modifications of the game form.[5] Kalai's convergence theorem shows that for any choice of ε greater than 0, the equilibria of the game approaches an ex post ε-Nash equilibrium as the number of players increases to infinity. Kalai's proof technique relies crucially on the use of finite type and action spaces, whereas our theorem holds for any compact subset of the finite-dimensional Euclidean space. Since many mechanism design techniques require connected type spaces, the use of uncountable type and action spaces is crucial.

Carmona [25] proves a series of results in spirit similar to our Theorem 2. However, as illustrated by example 3, proving that an exact equilibria of a nonatomic game is an epsilon equilibrium of a large finite game does not imply that the continuum game equilibrium is close, in either utility or strategy space, to the exact equilibria of the large finite game. Carmona's goal is to provide tools to analyze the set of equilibria of the continuum game by studying equilibria of large finite approximations and so this concern is not an issue. Our goal is precisely the opposite, to use the equilibria of simple continuum games as approximations for the equilibria of intractable game theoretic market models. Theorems 3 and 4, which have no analog in Carmona [25], are the crucial analytical tools we will use in our applications.

The literature on Walrasian strategic games addresses the question of whether finite games taken to represent strategic foundations for Walrasian equilibrium in fact achieve Walrasian equilibria in the limit as the number of agents increases (Roberts and Postlewaite [75], Otani and Sicilian [65], Jackson and Manelli [42]). Typically in these games the agents in the economy each declare a demand schedule, and the market price is defined by the price that clears the market given the aggregate declared demand. Jackson and Manelli provide the analyses closest in spirit to this paper. Jackson et al. prove that when agents believe they have little influence on the market-clearing price, then the agents' declared demands must converge to the Walrasian equilibrium demands as the economy grows in size. Our continuity condition, although based on the economic primitives, is similar to the equilibrium-dependent small influence condition of Jackson and Manelli.

[5]Given the use of finite action and type spaces, our continuity condition holds in every game considered in Kalai [43].

Mas-Colell and Vives [54] discuss implementation theory for exchange economies with a continuum of agents. The theoretical techniques employed are similar to ours in that they utilize approximation theorems to argue that implementations in continuum economies yield approximate Bayesian-Nash implementations in large finite economies. Our proof techniques, although similar to those in their paper, are more readily applied outside of the context of exchange economies to models such as large auctions for indivisible goods and matching markets. As noted above, our focus is not implementation per se but the analysis of existing large markets. Implementation problems similar in spirit to those tackled by Mas-Colell and Vives could be approached using our framework, although this issue is left for future work.

A growing literature employs the techniques of game-theoretic analysis to analyze the performance of large markets (Pesendorfer and Swinkels [69], Cripps and Swinkels [27], McLean and Postlewaite [51], and Fudenberg et al. [31] amongst many others). Most of the works in this literature proceed by studying the properties of game-theoretic equilibria as the number of agents increases to infinity. A notion of continuity, similar to the one we employ, underlies these results either explicitly or implicitly. In most of these models, the weak form of continuity our results require is an innocuous technical restriction and, if assumed explicitly, the arguments presented in these papers can be constructed from more general sets of economic primitives using simpler proof techniques. We explicitly demonstrate how the work of McLean and Postlewaite and Fudenberg et al. fits within our framework in Appendix B.

1.2. **Outline of Paper.** Section 2 develops the principal results relating large finite mechanisms to their nonatomic analogs. Section 3 uses our theorems to analyze the behavior of uniform price auctions with a large number of agents who are allowed to have complementary preferences for multiple units of a homogenous good. Section 4 analyzes conditions under which Markov perfect equilibria of dynamic models can be approximated by stationary equilibria. Section 5 concludes the study. All proofs have been relegated to Appendix A. Proofs omitted for length can be obtained from the author. Additional applications to McLean and Postlewaite's [51] model of informational smallness in partial information general equilibrium settings and the strategic smallness results of Fudenberg et al. [31] are available from the author.

2. Main Results

In this section we examine the relationship between large finite mechanisms and their nonatomic analogs and provide sufficient conditions for the equilibria of the nonatomic mechanisms to be approximate equilibria of the analogous large finite mechanisms and vice versa. In section 2.1 we define the economic primitives used in our mechanism design models and define the topologies required to prove our results. Section 2.2 presents results on the existence of Nash equilibria of nonatomic mechanisms in our setting. Section 2.3 describes the theorems we use to relate the equilibria of the large finite mechanisms and their nonatomic analogs. Finally, section 2.4 presents results that analyze the effect on mechanism continuity of adding noise to the outcome function.

2.1. **Model Framework.** Consider the probability space $(\Omega, \mathcal{F}, P)$ where Ω denotes a state space, $\mathcal{F}$ a σ-algebra of events on Ω, and P a probability measure on the space. We will denote the probability of an event E as $P(E)$ or $\Pr^{P}\{E\}$ where the superscript P refers to the measure determining the probability. The notation specifying the measure will be omitted when confusion will not result. Let $\Theta \subset \mathbb{R}^{d}$ denote a type space for the agents and $\Delta(\Theta)$ the space of measures over Θ. Each state $\omega \in \Omega$ defines the types for each of the agents in the large finite mechanisms we study.[6] The type space can denote agent preferences, private information, or agent beliefs. We will let $\theta \in \Theta$ and $\pi^{\Theta} \in \Delta(\Theta)$ denote generic elements from these spaces. Let $\mathcal{X}$ be the outcome space for the mechanism. We will assume that $\mathcal{X}$ is a metric space $(\mathcal{X}, d_{\mathcal{X}})$ and endow $\mathcal{X}$ with the topology generated by $d_{\mathcal{X}}$.[7]

Assume the agents have symmetric utility functions $u_N : \Theta \times \Delta(\Theta) \times \mathcal{X} \to \mathbb{R}$ in the N agent mechanism.[8] Agent utilities in the limit game will be denoted $u : \Theta \times \Delta(\Theta) \times \mathcal{X} \to \mathbb{R}$. We will assume that the agent preferences can be represented in expected utility form when considering stochastic outcomes, Bayesian-Nash equilibria, or equilibria in mixed strategies. By including the empirical distribution of agent types in the utility function, we allow for certain forms of interdependent valuations. For example, common-value utility functions for auctions can be specified by letting $u_N(\theta, \pi^{\Theta}, x) = E[v(\theta, x)|\pi^{\Theta}]$ where the expected utility is conditional on the (semi-anonymous) information revealed by the other agents' types in equilibrium. This construction can also be used to formulate expected utilities

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[6] Formally each ω denotes the types of a countable infinity of agents. This formulation is used so that we can take limited as $N \to \infty$.

[7] The assumption that $\mathcal{X}$ is a metric space makes several of the proofs more direct. In most cases of interest the outcome space is a subset of $\mathbb{R}^{d}$, $d < \infty$, making the assumption of metric and topology obvious. However, arguments utilizing the metric $d_{\mathcal{X}}$ could be formulated using general topological notions of continuity if desired.

[8] The restriction to a common utility function requires that agent heterogeneity be expressable in terms of the type space. In previous versions of this paper, the requirement of a common utility function was weakened by placing an equicontinuity restriction on the family of allowable utility functions (as opposed to continuity on the common utility function) and restricting the proof to revelation mechanisms.

conditional on forms of uncertainty that are correlated with the types of the other agents. For example, agent types in a general equilibrium economy can represent informative signals about the state of aggregate demand, which could impact the expected utility derived by the agents and firms in the model.

We will require an assumption to relate preferences in the large finite game with preferences in the limit game as $N \to \infty$. In addition, we will require continuity of the utility functions in the limit. These continuity restrictions follow from the usual assumption of a continuous preference relation underlying the utility function representation.

Assumption 1. *For all $(\theta, \pi^{\Theta}, x) \in \Theta \times \Delta(\Theta) \times \mathcal{X}$, $u_N(\theta, \pi^{\Theta}, x) \to u(\theta, \pi^{\Theta}, x)$ uniformly.*

We will assume that the agents choose actions from an abstract message space $\mathcal{M}$.[9] The outcome of the N-agent mechanism is then a function that maps the agents' messages into outcomes, $g_N : \mathcal{M}_1 \times \mathcal{M}_2 \times ... \to \mathcal{X}^{\mathbb{N}}$, where the subscript on $\mathcal{M}_i$ denotes the message submitted by agent i. In the case of revelation mechanisms, it suffices to let $\mathcal{M} = \Theta$.[10] The strategy space, denoted Σ, of the agents is the set of measurable maps from Θ into $\Delta(\mathcal{M})$. When required, we will analyze the metric space (Σ, d_{Σ}) where for $\sigma, \sigma' \in \Sigma$

$$d_{\Sigma}(\sigma, \sigma') = \sup_{\theta \in \Theta} |\sigma(\theta) - \sigma'(\theta)| \tag{2.1}$$

The definition of semi-anonymous below captures the notion that the mechanism does not privilege relationships or interactions between particular agents. Note that we still allow the type of a given agent to play a role in determining her outcome.

Definition 1. *An N-agent mechanism is **semi-anonymous** if the outcome function can be written as $g_N^1(\Delta(\mathcal{M}) \times \mathcal{M}) \times g_N^2(\Delta(\mathcal{M}) \times \mathcal{M}) \times ...$ where $g_N^i : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$ is individual i's outcome function. $\Delta(\mathcal{M})$ is taken to be the empirical distribution of messages declared by the agents.*[11]

While semi-anonymity is not without loss of generality, it allows for models incorporating different roles for the agents (ex: buyers and sellers), the possibility of non-participation (ex: voters and abstainers), and random participation. See Kalai [43] for examples useful

[9] We will assume the agents all have the same message space for notational ease. This assumption can be weakened, so long as the mechanism still respects the semi-anonymity property defined below.

[10] As a notational convention, a null outcome $\varnothing$ must be identified to indicate the outcome for agents $N+1, N+2, ...$ in the N-agent mechanism. However, the need for this notational convention will not be eliminated by our assumption of semi-anonymity.

[11] This notion was first used by Green [35]. Some form of anonymity is assumed in most papers studying large markets, including those of Sabourian [81], Al-Najjar et al [3], and Kalai [43].

for reformulating traditional mechanisms into this framework. We will assume in this study that the mechanisms are semi-anonymous unless otherwise noted.

Semi-anonymity must be assumed if agents are to become uniformly strategically small as $N \to \infty$. The following two examples provide instances where the semi-anonymity condition is violated.

Example 1. *(Monopoly Pricing) Consider a model wherein a monopolist interacts with a large number of small consumers. The monopolist commits to a price schedule as a function of realized aggregate demand, and the consumers respond by announcing individual demand schedules. In this case, the agents will have minimal ability to affect the aggregate demand announcement as $N \to \infty$, which captures the intuition that the consumers are strategically small in this market. However, since the monopolist's action enters each agents' optimal decision problem non-anonymously, the price schedule choice of the monopolist can influence the decisions of every consumer in the market. The monopolist will obviously not be small, even in the limit as the number of consumers approaches infinity.*

Example 2. *(Local Interactions in a Large Economy) Consider a stylized game wherein the agents are arranged in a circle and labeled with integers in increasing order as one progresses around the circle. Each agent chooses a mixed strategy $\sigma_i \in \Delta(\{a, b\})$. An agent of type $\theta = 1$ earns 1 util for each neighbor's action she matches and 0 otherwise, whereas an agent of type $\theta = -1$ earns 1 util for each neighbor's action she mismatches and 0 otherwise. Note that an individual agent's payoff is affected by the actions of a vanishing fraction of the agents in the economy in the limit. However, individual i's neighbor $i+1$ employs a strategy that depends on her beliefs about σ_{i+2}, agent $i+2$'s strategy is influenced by her beliefs about σ_{i+3}, ad infinitum. Therefore, the equilibrium outcome for each agent is affected by the individual choices of all of the other agents in the economy, albeit in an indirect fashion. Therefore, the actions of the agents in the economy will not become small in the infinite limit.*

Consider the family of N-agent semi-anonymous mechanisms that we denote $\mathcal{O} = \{g_N : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}\}_{n=1}^{\infty}$. A family of outcome functions is required as we implicitly encode market feasibility requirements into these mechanism outcome functions. For example, suppose we consider a N agent auction that allocates M goods. The outcome function g_N then instantiates the desired allocation rule of the M goods amongst the N bidders. If we consider a larger auction with $2N$ bidders and $M + k$ goods, the outcome function g_{2N} then contains the feasibility requirement that up to $M + k$ goods are allocated amongst the $2N$ bidders.

As in the case of the utility functions, we will require a definition of the outcome function in the limit game, which we denote $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$. The outcome functions of the large finite mechanisms are related to the nonatomic limit mechanism outcome function by the following assumption.

Assumption 2. *For all $(m, \pi^{\mathcal{M}}) \in \Delta(\mathcal{M}) \times \mathcal{M}$ we have $g_N(m, \pi^{\mathcal{M}}) \to g\ (m, \pi^{\mathcal{M}})$ uniformly.*

Where required, we will refer to a family of functions $\{g_m : \Delta(\mathcal{M}) \to \mathcal{X}\}_{m \in \mathcal{M}}$ where $g_m(\pi^{\mathcal{M}}) = g(\pi^{\mathcal{M}}, m)$. We will typically demand that $\{g_m : \Delta(\mathcal{M}) \to \mathcal{X}\}_{m \in \mathcal{M}}$ be uniformly equicontinuous, which is equivalent to demanding that g be uniformly continuous in $\Delta(\mathcal{M})$ and that this uniformity hold across $\mathcal{M}$. The stronger assumption that g be uniformly continuous implies the uniform equicontinuity of $\{g_m : \Delta(\mathcal{M}) \to \mathcal{X}\}_{m \in \mathcal{M}}$.

The proof of the main result requires the use of several topological concepts from the theory of spaces of measures. The Glivenko-Cantelli lemma implies that given an infinite succession of realizations of independent and identically distributed (i.i.d.) random variables, the resultant empirical cumulative distribution function (CDF) converges to the true CDF almost surely in the Kolmogorov metric for distributions over finite dimensional Euclidean spaces.[12]

Definition 2. *Consider two cumulative distribution functions F, G over the state space $\Omega \subset \mathbb{R}^d$. The **Kolmogorov (Uniform) metric** is $d_K(F, G) = \sup_{x \in \Omega} |F(x) - G(x)|$*

We employ the Glivenko-Cantelli lemma to show that the empirical distribution of i.i.d. agent types is well behaved in the asymptotic limit as the number of mechanism participants increases. Therefore, the Kolmogorov metric will be important when analyzing continuity of mechanism outcomes and agent utility with respect to the distribution of participant types.

Endow the space $\Delta(\mathcal{M})$ with the topology generated by the Kolmogorov metric and suppose agent types are drawn in an i.i.d. fashion from measure π_0^{Θ} over Θ.[13] Given a measurable, symmetric strategy for the agents, $m : \Theta \to \Delta(\mathcal{M})$, a distribution of agent

[12]Please see the appendix for a formal description of the relationship between the Kolmogorov Metric and the weak-* topology.

[13]Note that we could allow the agents to be independently but not identically drawn from a finite set of differing type spaces $\{\Theta_i\}_{i=1}^m$ with the joint type space defined as $\Theta = \times_{i=1}^m \Theta_i$. If we denote the number of agents drawn from the i^{th} type space as N_i, we require $\frac{N_i}{N} \to \beta_i > 0$ as $N \to \infty$. This is, from the perspective of the designers and the agents, a process where the agent types are drawn in a two stage process: (1) randomly choose the type space and (2) draw the type from the distribution associated with the chosen type space. Note that step (1) could be done deterministically. For example, the analyst may be concerned with models wherein the number of buyers and sellers are equal. In this case, buyers and

types $\pi^{\Theta} \in \Delta(\Theta)$ induces a distribution of messages that we denote $\pi^{\mathcal{M}}$. Formally we write

$$\text{For all } M \subset \mathcal{M},\ \pi^{\mathcal{M}}(M) = \int_{\Theta} 1\{m(\theta) \in M\}\pi^{\Theta}(d\theta) \tag{2.2}$$

where $1\{m(\theta) \in M\}$ is an indicator function for the event that the message declared is within the set M.[14]

Our results will use the following notion of an $\varepsilon-$equilibrium for mechanisms.

Definition 3. *Given an N-agent mechanism with outcome function $g(\Delta(\mathcal{M}) \times \mathcal{M})$ and utility function $u : \Theta \times \Delta(\Theta) \times \mathcal{X} \to \mathbb{R}$, the strategy $m : \Theta \to \Delta(\mathcal{M})$ is an* ***Ex Post ε-Nash Equilibrium*** *at state $\omega \in \Omega$ if for all agents i, all $m' \in \mathcal{M}$, and all $m^* \in$supp[$m(\theta_i)$] we have*

$$u(\theta_i, \pi^{\Theta}(\omega), g(\pi^{\mathcal{M}}(\omega), m^*)) + \varepsilon \geq u(\theta_i, \pi^{\Theta}(\omega), g(\pi^{\mathcal{M}}(\omega) + \frac{1}{N}[\delta_{m'} - \delta_{m^*}], m')) \tag{2.3}$$

where $\pi^{\Theta}(\omega)$ is the empirical measure of types and $\pi^{\mathcal{M}}(\omega)$ is the empirical measure over the space of message in state $\omega \in \Omega$, and δ_m is an atomic measure placing weight 1 on m.

It is obvious from the definition that an ex post $\varepsilon-$Nash Equilibrium across almost all states $\omega \in \Omega$ is also an interim ε-Bayesian Nash Nash equilibrium.

2.2. **Nonatomic Models.**

We will define a non-atomic mechanism to describe the limit equilibrium of interest in our analysis.[15]

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sellers are defined as separate type space and enter the economy in (buyer, seller) pairs with types for each determined independently.

[14]Note that we are assuming M is a measurable set and that m is a measurable function under π^{Θ}.

[15]Several authors have independently developed related notions. For example, Budish's *continuum replication* [24] assumes that each agent is replaced by a unit mass of nonatomic agents with the outcome of this game defining the limit equilibrium. In our case we consider the limiting non-atomic game rather than a replication of the original preference structure of the mechanisms. Given an K agent preference profile $\{u_i\}_{i=1}^{K}$, we can consider continuum replications of this form by considering the measure that places mass $\frac{1}{K}$ on each element of the preference profile and take the limit as $N \to \infty$. Therefore, we see that this replication technique is a special case of the more general framework developed herein.

Definition 4. *An $\varepsilon-$Nash equilibrium message strategy $m^\infty : \Theta \to \Delta(\mathcal{M})$ of the nonatomic mechanism defined by utility function $u : \Theta \times \Delta(\Theta) \times \mathcal{X} \to \mathbb{R}$ and outcome function $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$ satisfies*

(2.4)
$$u(\theta, \pi_0^\Theta, g(\pi_0^\mathcal{M}, m) + \varepsilon \geq u(\theta, \pi_0^\Theta, g(\pi_0^\mathcal{M}, m') \text{ for all } m \in supp[m^\infty(\theta)], m' \in \mathcal{M}, \theta \in \Theta$$

where $\pi_0^\mathcal{M}$ is the distribution of messages induced by the strategy m^∞ defined by

$$\text{For all } M \subset \mathcal{M},\ \pi_0^\mathcal{M}(M) = \int_\Theta 1\{m^\infty(\theta) \in M\}\pi_0^\Theta(d\theta) \tag{2.5}$$

We use the notation supp$[\pi^\Theta]$ to refer to the support of a measure π^Θ. Note that deviations from the messaging strategy by a single agent do not affect the aggregate distribution of messages $\pi_0^\mathcal{M}$. Therefore, in the nonatomic game, the choice of message by a single agent has no effect on other agents' outcomes.

Denote the *extended type space* as $\Theta^E = \Theta \times [0,1]$ and assume that types are drawn according to the product measure $\pi_o^\Theta \times \lambda(\circ)$ where λ refers to the Lebesgue measure on $[0,1]$. Since for $E \subset \Theta$ we have $\pi_o^\Theta(E) = \pi_o^\Theta \times \lambda(E \times [0,1])$, the marginal distribution of types over Θ is the same in both spaces.[16]

Theorem 1. *Suppose $u(\theta, \pi_0^\Theta, g(\circ,\circ)) : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathbb{R}$ is continuous and $\mathcal{M}$ is compact. Then there exists a symmetric pure strategy Nash Equilibrium in the nonatomic version of the mechanism in the extended type space.*

The proof for this theorem relies on Mas-Colell [53], which showed that there exists an equilibrium distribution over the product space of types and messages such that almost all type-message pairs represent an optimal response to the equilibrium distribution. Due to our assumption that type and message spaces are subsets of the finite dimensional Euclidean space, given such an equilibrium distribution we can define probability density functions over the space of messages conditional on each type. We interpret this conditional distribution as a symmetric mixed strategy. By using the extended type space, we can then purify the mixed strategy to a symmetric pure strategy in the extended type space.[17] This theorem is useful in that it establishes that any nonatomic mechanism in our setting possesses a Nash equilibrium, which provides assurance that our nonatomic approximation is not vacuous.

[16]The extension of the type space to Θ^E is a modeling technique that does not change the empirical content of the model. In effect, it provides a method for purifying a nonatomic mixed strategy equilibrium.

[17]The existence of a symmetric pure strategy equilibrium in the original type space has only been proven for special cases, most notably finite action spaces (see [53] and [84]).

2.3. **Relation Between Large Finite and Nonatomic Economies.** The primary theorems of this paper analyze the relationship between the equilibria of the nonatomic mechanism and the equilibria of the large finite mechanism. The archetypal feature of nonatomic mechanisms is that agents cannot affect the incentives of the other agents by deviating from the equilibrium strategy. For large markets, agents have at most a small effect on the empirical CDF of realized actions with this effect vanishing as the market grows. Intuitively, given that the appropriate notions of continuity hold, the agents' small effect on the empirical CDF of messages will have only a small effect on the outcome for other agents. The formalization of this argument is the following theorem.

Theorem 2. *Assume $\mathcal{M} \subset \mathbb{R}^d$ and fix $\varepsilon > 0$ and $\rho \in [0, 1)$. For each equilibrium $m^\infty : \Theta \to \Delta(\mathcal{M})$ of the nonatomic version of the semi-anonymous mechanism $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$ there exists an N^* such that for all $N > N^*$, $m^\infty(\circ)$ is with probability $1 - \rho$ an ex post ε-Nash equilibrium of the N-agent mechanism $g_N : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$* ***if***

(1) *The family $\{g(m, \cdot)\}_{m \in \mathcal{M}}$ is uniformly equicontinuous in some relatively open set $\Delta(\mathcal{M})$ containing $\pi_0^{\mathcal{M}}$.*[18]
(2) *The family $\{u(\theta, \cdot, \cdot)\}_{\theta \in \Theta}$ is uniformly equicontinuous in the space $[\{\theta\} \times \Delta(\Theta) \times \mathcal{X}] \cap \mathcal{S}$ where*

$$\mathcal{S} \supset \{(\theta, \pi_0^{\Theta}, x) \in \Theta \times \Delta(\Theta) \times \mathcal{X} : g(\pi_0^{\mathcal{M}}, m^\infty(\theta) = x\}$$

is a relatively open set in the product topology on $\Theta \times \Delta(\Theta) \times \mathcal{X}$

The intuition for the proof relies on combining our continuity conditions to show that the optimization problem facing an agent in a large finite mechanism is nearly identical to the decision problem of an agent in a nonatomic mechanism. Since the agent types represent i.i.d. draws from the true distribution π_0^{Θ}, we show that in the large finite economy, the empirical CDF of agent types almost surely converges to π_0^{Θ} in the Kolmogorov metric as $N \to \infty$. If the agents follow the nonatomic equilibrium strategy m^∞, then the mechanism outcomes for each type will be inside $\mathcal{S}$ for large enough N. Note that $\mathcal{S}$ contains the set of type-distribution-outcome triples realized in the equilibrium of the nonatomic mechanism. Since the mechanism and utility functions are continuous within $\mathcal{S}$, the problem facing the agents in the large finite mechanism will be approximately the same as the one facing the agents in the nonatomic mechanism.[19] Therefore, m^∞ will be an ε-equilibrium of the

[18] Note that uniform continuity of $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$ in a relatively open set containing $\{\pi_0^{\mathcal{M}}\} \times \mathcal{M}$ suffices for condition (1). Similarly, uniform continuity of u over a relatively open set containing $\mathcal{S}$ suffices for condition (2).

[19] Although we derive our result from the model primitives, an alternative proof employs the Glivenko-Cantelli lemma to show that the problem facing the agents will lie in $\mathcal{S}$ with high probability for sufficiently

large finite mechanism. Since this argument applies almost surely over Ω in the asymptotic limit, we can choose a large enough N so that this argument holds for a measure $1-\rho$ of the state space for any ρ less than 1. Therefore, with probability $1-\rho$ the ε-equilibrium will be an ex post $\varepsilon-$Nash equilibrium.

When the number of agents grows, the individual agents can make sharp predictions about the distribution of the types participating in the mechanism. Therefore, if the mechanism is continuous, the agents can ex ante make precise predictions about the effect of their message on their ex post outcome. The outcome of mechanisms with incomplete information is thus well approximated by the nonatomic, complete information analogs as the number of agents approaches infinity. This implies that the equilibrium will be robust to a wide variety of assumptions regarding the knowledge structure encapsulated in the type space utilized in the model.[20]

The notion of an ex post ε-Nash Equilibrium with high probability provides useful insights into the likely outcome of markets when the number of agents is large, but it is of interest to link this equilibrium notion to more traditional notions of market equilibrium for large finite mechanisms. Fortunately, if we impose an innocuous restriction on the utility functions we can show that our ex post ε-Nash Equilibrium implies the more traditional notion of a Bayesian Nash Equilibrium.

Corollary 1. *Assume the conditions for Theorem 2 hold. Further, assume that the utility functions of the agents are bounded over the range of the outcome function* $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$. *Then for any choice of* $\varepsilon > 0$ *and each equilibrium* $m^{\infty} : \Theta^E \to \mathcal{M}$ *of the nonatomic version of the mechanism there exists and* N^* *such that for all* $N > N^*$, m^{∞} *is an* $\varepsilon-$*Bayesian Nash equilibrium of the* N *agent version of the mechanism.*

The proof for this corollary is straightforward and omitted from this study. For any choice of $\rho \in (0,1]$ there exists a number of agents N^* such that if $N > N^*$ participate in the mechanism, m^{∞} will dictate an action within ε of the optima with probability at least $1-\rho$. Given the boundedness of the agent utility functions over the set of possible mechanism outcomes, the remaining measure ρ outcomes will yield an outcome of at most $M < \infty$ less than the optima. Therefore the benefit of deviating from m^{∞} at the interim stage is at most $\varepsilon + \rho * M$. Since for N^* sufficiently large ρ and ε can be chosen to be arbitrarily small, m^{∞} is an ε-Bayesian Nash equilibrium for any choice of $\varepsilon > 0$ for sufficiently large, but finite, choices of N^*.

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large N^* with Berge's Theorem of the Maximum to establish continuity of the value function within the neighborhood of $\mathcal{S}$.

[20] See Bergmann and Morris [19] for details on ex post equilibria and belief hierarchies

The theorem above separately imposes continuity conditions on the outcome function g and the utility function u. A weaker continuity condition, but still sufficient for the application of our result, is

Assumption 3. *Consider a fixed nonatomic equilibrium strategy m^∞ and a fixed choice of a relatively open set $\mathcal{S}$ such that*

$$\mathcal{S} \supset \{(\theta, \pi_0^\Theta, m, \pi_0^\mathcal{M}) \in \Theta \times \Delta(\Theta) \times \mathcal{M} \times \Delta(\mathcal{M}) : m^\infty(\theta) = m\}$$

Assume that the composite function $u(\theta, \pi^\Theta, g(m^\infty, \pi^\mathcal{M}))$ mapping from $\Theta \times \Delta(\Theta) \times \mathcal{M} \times \Delta(\mathcal{M})$ to $\mathbb{R}$ is equicontinuous for the family of sets of the form $[\{\theta\} \times \Delta(\Theta) \times \{m\} \times \Delta(\mathcal{M})] \cap \mathcal{S}$

This alternative assumption is weaker in that it allows discontinuities in the mechanism where these discontinuities do not affect the utility reaped by any type of agent in the economy. For most of the analyses presented below, both forms of continuity hold (or fail to hold) jointly.

The theorem above is stated in terms of ex post ε-Nash equilibria. However, it is straightforward to prove an analogous result for deviations by fixed finite coalitions of agents within the economy. In essence, what we will show is that for any coalitions with K or fewer agents, there is a large enough N that the outcome will be an ex post K-Coalition-Proof ε-Nash equilibrium.

Corollary 2. *Consider any fixed finite set of K agents in the economy denoted with indices $\mathcal{I} = \{i_1, ..., i_K\} \subset \mathbb{N}$. Given the conditions of theorem 2 hold, for any $\varepsilon > 0, \rho \in (0,1]$ there is an $N \geq i_K$ that that for all $i \in \mathcal{I}$ the following holds with probability $1 - \rho$*

$$u(\theta_i, \pi^\Theta, g(\pi^\mathcal{M}, m(\theta_i))) + \varepsilon \geq \max_{m'=(m'_1,...,m'_K)\in\mathcal{M}^K} u(\theta_i, \pi^\Theta, g(\pi^\mathcal{M} + \tfrac{1}{N}\textstyle\sum_{i=1}^K [\delta_{m'_i} - \delta_{m(\theta_i)}], m'_i))$$

Intuitively any finite set of deviations has a vanishingly small impact on the $\Delta(\mathcal{M})$ term for large N. This implies that finite groups of agents cannot collude to alter the outcome of the mechanism. Corollary 2 can be viewed as a coalition-proofness refinement for mechanism stability. For the purposes of market design and real-world mechanism analysis, collusion by a small group of agents within a large market is often feasible, and the analyst can draw comfort that under the conditions outlined above she need not worry about impact of this form of deviation. Our result would not generally hold if a non-vanishing fraction of the agents collude, but markets are often designed because market-wide coordination is infeasible.

Theorem 2 shows that the set of nonatomic equilibria is a subset of the set of ex post ε-Pure Strategy Nash equilibria of the large finite mechanism in the limit as $N \to \infty$. We can show that all of the equilibria of the large finite mechanism have nonatomic analogs, in the sense that equilibria of the large finite mechanism are epsilon equilibria of the corresponding nonatomic mechanism.

Theorem 3. *Assume $\mathcal{M} \subset \mathbb{R}^d$ and fix $\varepsilon > 0$. There exists an N^* such that for all $N > N^*$ and for any symmetric Bayesian Nash equilibrium $m^N : \Theta \to \Delta(\mathcal{M})$ of the N agent mechanism, the finite agent strategy m^N is an ε-Nash equilibrium of the nonatomic mechanism* ***if***

(1) *The family $\{g(m, \cdot)\}_{m \in \mathcal{M}}$ is uniformly equicontinuous in some relatively open set $\Delta(\mathcal{M})$ containing π_∞^N where $\pi_\infty^N(M) = \int_\Theta \Pr\{m^N(\theta) \in M\} \pi_0^\Theta(d\theta)$.*
(2) *The family $\{u(\theta, \cdot, \cdot)\}_{\theta \in \Theta}$ is uniformly equicontinuous and bounded in the space $[\{\theta\} \times \Delta(\Theta) \times \mathcal{X}] \cap \mathcal{S}$ where $\mathcal{S} \supset \{(\theta, \pi_0^\Theta, x) \in \Theta \times \Delta(\Theta) \times \mathcal{X} : g(\pi_\infty^{\mathcal{M}}, m^N(\theta^E)) = x\}$ is a relatively open set in the product topology on $\Theta \times \Delta(\Theta) \times \mathcal{X}$*

The principle difficulty in proving this theorem is to show that as N goes to infinity, the optimization problem facing the agents is dominated by the high probability event that the distribution of messages received by the mechanism is close to π_∞^N in the Kolmogorov metric. Corollary 5 demonstrates a uniform bound across measures on the probability that the empirical CDF will be more than δ in distance from π_0^Θ in the Kolmogorov metric, which we denote as event E, and that this probability approaches 0 as $N \to \infty$. Since the utility functions are bounded, the difference in utility between the best and worst message contingent on E occurring is bounded. For N sufficiently large, the probability of E can be made arbitrarily small. Therefore, for any choice of $\delta > 0$, the incentive for an agent to consider states of the world wherein the empirical and true measure differ by more than δ vanishes as $N \to \infty$.

Since the mechanism and utility function are continuous in the Kolmogorov metric, if the empirical CDF and π_0^Θ are sufficiently close (the event $\Omega \backslash E$) and N sufficiently large, then the agents' optimization problem in the large finite mechanism is approximately the same as that facing the agents in the nonatomic mechanism. This implies that for sufficiently large N, a solution to the problem facing agents in the large finite mechanism will approximately solve the optimization problem for agents in the nonatomic mechanism when all other nonatomic agents follow m^N. Therefore, m^N will be an ε-Nash equilibrium of the nonatomic mechanism.

Let the set of exact Bayesian-Nash equilibria of the N agent game be denoted by the correspondence $\mathcal{E} : \mathbb{N} \rightrightarrows \Sigma$. Let $\mathcal{E}^\infty = \underset{N \to \infty}{Lim}\, \mathcal{E}(N)$ and denote the set of equilibria of the nonatomic analog as $\mathcal{E}^{NA}$. We can use Theorem 3 to prove the following relationship between the exact equilibria of the large finite model and the exact equilibria of the nonatomic model. Note that unlike the results above, Theorem 4 implies convergence of the equilibria in the strategy space, whereas Theorems 2 and 3 suggest convergence of equilibria only in utility space.[21]

Theorem 4. *Assume* $\mathcal{M} \subset \mathbb{R}^d$. *The correspondence* $\mathcal{E}$ *is upper hemicontinuous with* $\underset{N\to\infty}{Lim}\, \mathcal{E}(N) = \mathcal{E}^{\infty} \subset \mathcal{E}^{NA}$ ***if***

1. *The family* $\{g(m,\cdot)\}_{m\in\mathcal{M}}$ *is uniformly equicontinuous over* $\Delta(\mathcal{M})$.
2. *The family* $\{u(\theta,\cdot,\cdot)\}_{\theta\in\Theta}$ *is uniformly equicontinuous over* $\{\theta\} \times \Delta(\Theta) \times \mathcal{X}$.
3. $u(\theta, \pi_0^{\Theta}, g(\cdot,\cdot))$ *is upper semi-continuous in* $\Delta(\mathcal{M}) \times \mathcal{M}$ *where* $\Delta(\mathcal{M})$ *is endowed with the weak-* topology*

The above theorem implies that any convergent sequence of exact equilibria of the large finite game converges to some equilibrium of the nonatomic model. Suppose such a convergent sequence did not have a limit point in $\mathcal{E}^{NA}$. This would imply that if the agents in the nonatomic game played the limit strategy, some positive measure of these agents would have a profitable deviation. The statement of Theorem 3 strongly hints at a contradiction, and our proof goes through the formal steps of showing this to be the case.[22]

In order to make our intuition precise, we need to strengthen our continuity assumptions. Consider an arbitrary exact equilibrium of the N agent game, $m^N : \Theta \to \Delta(\mathcal{M})$. We then know that if all of the agents follow m^N in the continuum game, we will realize a distribution

$$\pi_{\infty}^{N}(M) = \int_{\Theta} \Pr\{m^N(\theta) \in M\} \pi_0^{\Theta}(d\theta) \tag{2.6}$$

[21] The interpretation of Theorems 2 and 3 as convergence in utility space is merely suggestive. For example, we proved that an equilibrium of the nonatomic game was an ε-equilibrium of the large finite game. This suggests that the nonatomic game equilibrium is an "approximation" of some large finite equilibrium in utility space. However, there are examples of games whose exact equilibria of generic games can yield payoffs that are non-trivially different than any ε-equilibrium for any $\varepsilon > 0$ (for an early example, see Roberts and Postlewaite [75]). Theorem 4 shows that such an example cannot be generated if our semianonymity and our continuity restrictions hold.

[22] We could weaken the conditions of our theorems to the following, although we choose not to do so for expositional ease:
Suppose the following two conditions hold for all $m^N \in \mathcal{E}(N)$ for all $N \in \mathbb{N}$.

1. The family of functions $\{g_m : \Delta(\mathcal{M}) \to \mathcal{X}\}_{m\in\mathcal{M}}$ is uniformly equicontinuous in some relatively open set $\Delta(\mathcal{M})$ containing π_{∞}^{N} where $\pi_{\infty}^{N}(M) = \int_{\Theta} \Pr\{m^N(\theta) \in M\}\pi_0^{\Theta}(d\theta)$.
2. The family of functions $\{u_\theta : \Delta(\Theta) \times \mathcal{X} \to \mathbb{R}\}_{\theta\in\Theta}$ is uniformly equicontinuous and bounded in the space $[\{\theta\} \times \Delta(\Theta) \times \mathcal{X}] \cap \mathcal{S}$ where $\mathcal{S} \supset \{(\theta, \pi_0^{\Theta}, x) \in \Theta \times \Delta(\Theta) \times \mathcal{X} : g(\pi_{\infty}^{\mathcal{M}}, m^N(\theta^E)) = x\}$ is a relatively open set in the product topology on $\Theta \times \Delta(\Theta) \times \mathcal{X}$

Unfortunately, even if there is some equilibrium of the nonatomic game, $m^\infty : \Theta \to \Delta(\mathcal{M})$, such that

$$\sup_{\theta\in\Theta}|m^\infty(\theta) - m^N(\theta)| < \delta \tag{2.7}$$

it is not the case that the distribution

$$\pi_0^{\mathcal{M}}(M) = \int_\Theta 1\{m^\infty(\theta) \in M\}\pi_0^\Theta(d\theta) \tag{2.8}$$

is close to π_∞^N in the Kolmogorov topology, although the two measures are close in the weak-* topology. The Kolmogorov topology registers small changes when individuals deviate in the nonatomic game, but a large change if a positive measure of agents deviates even a small amount. The weak-* topology is continuous to this form of change in measure, and hence we require this stronger notion of continuity in our proof. We discuss techniques sufficient for conditions (1) and (2) of Theorem 4 to be sufficient for condition (3) in Appendix B from the author.

Two notes are warranted at this point, our notion of approximate equilibrium in the nonatomic limit game requires that the agents uniformly choose approximately optimal actions. If we had weakened the notion of ε-Nash equilibrium to mandate a measure $1 - \varepsilon$ og agents choose actions within ε of the optimal choice, we would have recovered upper hemicontinuity of $\mathcal{E}$, but only if Σ is endowed with the L_0 norm. Our stronger notion of approximate nonatomic equilibria allows us to prove the stronger notion of upper hemicontinuity above.

Second, our continuity conditions are stronger than required for our proof techniques. Much as in Theorem 3,we could weaken our continuity conditions to hold only in a small set of the message distributions and outcomes induced by $\mathcal{E}^\infty$. However, without explicitly computing the equilibria of the large finite game and deriving the limit set $\mathcal{E}^\infty$, we cannot be assured where continuity of the nonatomic analog is required. As computation of large finite equilibria is precisely the step that we are trying to avoid using our analysis framework, it is difficult in practice to weaken the continuity conditions unless subsets of the strategy space can be ruled out of consideration as equilibria prior to equilbirium computation.[23]

We can interpret Theorem 4 explicitly in terms of the metric convergence of sequences of exact equilibria of the large finite games to equilibria of the nonatomic games. For the notion of distance in the message space to have meaning, we will assume that the

[23]The possibility of ruling out strategies prior to computing an equilibrium is suggestive of the exclusion of non-rationalizable strategies.

message space, $\mathcal{M}$, is a subset of a metric space with metric $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{+}$. Given the metric space $(\mathcal{M}, d_{\mathcal{M}})$, denote the associated metrization of the weak-* topology over $\Delta(\mathcal{M})$, known as the Lévy-Prokhorov metric, as $(\Delta(\mathcal{M}), d_{LP})$. As the following corollary follows directly from Theorem 4, we omit the proof.

Corollary 3. *Assume $\mathcal{M} \subset \mathbb{R}^d$ and fix $\delta > 0$. There exists an N^* such that for all $N > N^*$ and for any symmetric Bayesian Nash equilibrium $m^N : \Theta \to \Delta(\mathcal{M})$ of the N agent mechanism, there exists an equilibrium of the nonatomic mechanism $m^\infty : \Theta \to \Delta(\mathcal{M})$ such that for all $\theta \in \Theta$, $d_{LP}(m^N(\theta), m^\infty(\theta)) < \delta$* ***if***

(1) *The family $\{g(m, \cdot)\}_{m \in \mathcal{M}}$ is uniformly equicontinuous*
(2) *Assume the family of functions $\{u(\theta, \cdot, \cdot)\}_{\theta \in \Theta}$ is uniformly equicontinuous*

Theorem 5. (1) *$u(\theta, \pi_0^\Theta, g(\cdot, \cdot))$ is upper semi-continuous in $\Delta(\mathcal{M}) \times \mathcal{M}$ where $\Delta(\mathcal{M})$ is endowed with the weak-* topology*

The convergence rate result of theorem 14[24] t implies that the ε-best response results of theorems 2 and 3 can be chosen so that $\varepsilon = O(\frac{1}{\sqrt{N}})$. Unfortunately, similar asymptotic convergence rates cannot be provided for the strategy space approximation result of theorem 4 without additional assumption on the agents' objective function. One possible assumption is:

Assumption 4. *Consider any symmetric Bayesian Nash equilibrium $m^N : \Theta \to \Delta(\mathcal{M})$ of the N agent mechanism. Let*

$$S(\theta) = \arg\max_{m \in \mathcal{M}} u(\theta_i, \pi_\infty^\Theta, g(\pi_\infty^N, m)) \tag{2.9}$$

where

$$\pi_\infty^N(M) = \int_\Theta \Pr\{m^N(\theta) \in M\} \pi_0^\Theta(d\theta). \tag{2.10}$$

Let $\alpha(m;\theta) = \inf_{m' \in S(\theta)} d(m, m')$. Suppose that there exists a function $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that for some $\delta > 0$, if $m^* \in S(\theta)$ and $\alpha(m;\theta) < \delta$, then

$$|u(\theta_i, \pi_\infty^\Theta, g(\pi_\infty^N, m)) - u(\theta_i, \pi_\infty^\Theta, g(\pi_\infty^N, m^*))| > b(\alpha(m;\theta)) \tag{2.11}$$

Further assume that b is invertible in a neighborhood of 0.

The assumption above mandates that as we consider messages farther and farther outside of the set of optimal messages, the utility loss of these suboptimal messages must grow at

[24] Please see the Appendix for a formal statement and proof of the convergence rate result.

a rate of at least $b(\circ)$. If we combine our assumptions on $b(\circ)$ with the convergence rate result of theorem 14,[25] then it must be the case that theorem 4 can be strengthened to the claim that $\delta = O(b^{-1}(\frac{1}{\sqrt{N}}))$.

Ideally, we would like to prove that the limit of the sets of equilibria of the large finite mechanisms is lower hemicontinuous as N goes to infinity. However, as the following example shows, this is not the case for arbitrary mechanisms.

Example 3. *Consider a mechanism with outcome function $g(\pi^\Theta, m) = E^{\pi^\Theta} m$ and message space $\mathcal{M} = [0,1]$. The agents each name a number in $[0,1]$ and the mechanism pays each agent the average of the agents' announcements. Assume the agents have utility functions that are independent of type and increasing in monetary payments. Obviously for any finite mechanism the unique equilibrium is for each agent to announce the message $m = 1$ and receive a payment of 1. However, any messaging strategy is an equilibrium of the nonatomic mechanism as the individual agents cannot use their messages to affect the average computed by the mechanism. This does not disprove Theorem 3 since in the finite mechanism with $N > \frac{1}{\varepsilon}$ agents, any messaging strategy is an ε-equilibrium.*

The proof technique we used to show that all of the equilibria of the finite mechanism converge in strategy space to equilibria of the nonatomic mechanism cannot be used to prove the converse since Anderson's almost implies near theorem needs to be applied independently to each finite mechanism as $N \to \infty$. The example above reveals that for arbitrary mechanisms, we cannot choose an $\varepsilon > 0$ uniformly across N so that the equilibria of the finite mechanisms are within $\delta > 0$ of the nonatomic equilibria for arbitrary choices of δ.

In order to make predictions about the equilibrium strategies of the finite mechanism using the nonatomic equilibria as an approximation, it is crucial that the set of equilibria of the nonatomic mechanism be small relative to the size of the strategy space or that the equilibrium outcome of interest be fixed across the set of nonatomic equilibria. In the example illustrated above the set of nonatomic equilibria is equal to the entire strategy space, so no conclusions about the behavior of the agents in the exact equilibrium of the finite mechanism can be made using our analysis framework

2.4. **Mechanism Noise.** The continuity of the mechanism $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$ is crucial for the equilibria of the nonatomic mechanism to be approximate equilibria of the large finite mechanism (and vice versa). Several prior works have suggested that adding noise to a mechanism's outcome function (for example Swinkels [87], Levine et al. [49]) can restore

[25] Please see the Appendix for a formal statement and proof of the convergence rate result.

continuity to an otherwise discontinuous mechanism. As we will see in our uniform price auction application in Section 5, adding an arbitrarily small amount noise to the outcome of an auction can restore continuity when the deterministic auction is discontinuous. We will analyze model alterations of this form and provide sufficient conditions for noise to render a mechanism continuous.

Assume $\mathcal{M} \subset \mathbb{R}^d$ is a compact set for $d < \infty$. Endow $\mathcal{M}$ with the usual topology, $\Delta(\mathcal{M})$ with the weak-* topology, and $\Delta(\mathcal{M}) \times \mathcal{M}$ with the corresponding product topology. Let $\mathcal{X}$ be endowed with an analytically convenient topology. Consider a family of mechanisms $\mathcal{G} = \{g_\alpha : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}\}_{\alpha \in \Lambda}$ where Λ can be described as a probability space $(\Lambda, \mathcal{Q}, Q)$. Then one can define the random variable $g : \Lambda \times \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}$ where $g(\alpha, \pi^{\mathcal{M}}, m) = g_\alpha(\pi^{\mathcal{M}}, m)$ is assumed to be measurable under the product measure space $(\Omega \times \Lambda, \mathcal{F} \times \mathcal{Q}, P \times Q)$. This can be interpreted as a draw from Ω at the interim stage to determine the agent's types, followed by a draw from Λ at the ex post stage to determine which element of $\mathcal{G}$ generates the agents' outcomes. Therefore, the agents face a mechanism of the form $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \Delta(\mathcal{X})$ at the interim stage, where $\Delta(\mathcal{X})$ reflects the interim uncertainty as to which mechanism will be realized ex post. We endow $\Delta(\mathcal{X})$ with the weak-* topology.

A common problem facing a mechanism designer is a family of mechanisms $\mathcal{G} = \{g_\alpha : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}\}_{\alpha \in \Lambda}$ wherein each element $g_\alpha \in \mathcal{G}$ is discontinuous in such a fashion that makes application of the theorems above difficult. As we will see in section 3, a nonatomic, multi-unit auction mechanism with a deterministic supply will have a discontinuity for agent valuation declarations on the margin between provision and non-provision of the good. However, the location of this discontinuity is different for each value of the supply. This suggests that if the supply is random, the discontinuity in the individual mechanisms g_α will be smoothed by the randomization at the interim stage. Formally we define a family $\mathcal{G}$ of auctions with differing supplies, and the supply level (element of $\mathcal{G}$) used to determine allocations is determined ex post and is unknown at the interim stage when agents declare bids.

In order for randomizing over discontinuous mechanisms to yield a stochastic mechanism continuous at a message distribution $\pi_0^{\mathcal{M}}$, we will require that the discontinuities not be concentrated in the neighborhood system of $\pi_0^{\mathcal{M}}$.

Definition 5. *Consider the neighborhood system of $\pi_0^{\mathcal{M}}$ generated by d_K, denoted $\mathcal{N}(\pi_0^{\mathcal{M}})$. Consider an arbitrary $m \in \mathcal{M}$ and an arbitrary sequence $\{\mathcal{U}_n\}_{n=1}^\infty, \mathcal{U}_n \in \mathcal{N}(\pi_0^{\mathcal{M}})$ such that $i > j$ implies $\mathcal{U}_i \subset \mathcal{U}_j$. The discontinuities of $\mathcal{G} = \{g_\alpha : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}\}_{\alpha \in \Lambda}$ are* ***diffuse*** *around $\pi_0^{\mathcal{M}}$ if for any $m \in \mathcal{M}$ the elements of the sequence $q_n = Q(\{\alpha \in \Lambda : g_\alpha(\cdot, m)$*

is discontinuous over $\mathcal{U}_n$}) are well defined and $q_n \to 0$ as $n \to \infty$. Further, for the measure $1 - q_n$ of members of $\mathcal{G}$ that are continuous over $\mathcal{U}_n$, we will assume these are equicontinuous over $\mathcal{U}_n$.

It is clear from the definition of the sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ that the values of q_n must be decreasing as $n \to \infty$. The content in the definition lies in the assumption that q_n approaches 0 as n diverges to infinity and that for those elements of $\mathcal{G}$ that are continuous over $\mathcal{U}_n$ we can assume equicontinuity rather than simple continuity. This implies that for any $\varepsilon > 0$ we can choose a neighborhood of $\pi_0^{\mathcal{M}}$ such that all but but a measure ε of the members of $\mathcal{G}$ are continuous in the neighborhood. Assuming the discontinuities are bounded, then the impact these discontinuous mechanisms have on the continuity of the stochastic mechanism will fade as $\varepsilon \to 0$. This intuition underlies the following theorem.[26]

Theorem 6. *Suppose the outcome set $\mathcal{X}$ is bounded. Consider a family of mechanism $\mathcal{G} = \{g_\alpha : \Delta(\mathcal{M}) \times \mathcal{M} \to \mathcal{X}\}_{\alpha \in \Lambda}$ where the discontinuities of $\mathcal{G}$ are diffuse for all $\pi^{\mathcal{M}} \in \Delta(\mathcal{M})$. Then the stochastic mechanism $g : \Delta(\mathcal{M}) \times \mathcal{M} \to \Delta(\mathcal{X})$ is continuous in $\Delta(\mathcal{M})$ for fixed $m \in \mathcal{M}$ at the interim stage.*

Two technical issues about our formulation of smoothing mechanisms through added noise require discussion. First, it is important to note that the randomization is over an aggregate parameter. A unique mechanism outcome function is used at the ex post stage to generate outcomes for all of the agents. This contrasts sharply with the case of independent, identically distributed shocks observed privately by each agent. To see this, note that this private information can be encoded as an element of each agent's privately known type. In the limit as N approaches infinity, this type distribution will approach π_o^{Θ}, implying that the idiosyncratic noise vanishes in the aggregate. Second, we should note that the uncertainty is resolved at the ex post stage after all of the agents have issued their message declarations. If the uncertainty were resolved at the interim stage, the agents could condition their messages on the particular discontinuous mechanism they faced, which would render our theorems inapplicable to the economy.

3. Large Auctions with Complementary Preferences

Since the resurgence of interest in large scale auctions, abetted significantly by the 1994 Federal Communications Commission Spectrum Auctions, the vast majority of allocation mechanisms implemented have been uniform price auctions. It is well known that uniform

[26]A trivial corollary of this theorem is that randomizing over a family of continuous mechanisms preserves the continuity of the stochastic mechanism.

price auctions where agents have preferences for multiple goods are not generically efficient since agents have an incentive to withhold their demand for marginal units in order to drive down the price paid for the inframarginal units received (Back and Zender [13], Ausubel and Crampton [11]). Despite these disappointing theoretical properties, uniform price auctions have being widely implemented in both dynamic and static forms to allocate goods ranging from United States Treasury Bills to electricity provision rights.

Prior studies in the uniform price auction literature (for example Swinkels [87]) have employed restrictions on the model, such as insisting the agents are risk neutral and have decreasing marginal valuations for the goods, in order to apply game-theoretic analysis techniques to discover the equilibrium strategies and outcomes in the limit as $N \to \infty$. In the context of electricity auctions, it is clear that the right to provide successive units of power to an electricity good could be complements to each other. Consider an electricity generator with a large fixed but low marginal cost of production. The average cost for the generator will be declining as the units provided increases, which means the profit function would have the complements property with respect to the number of units demanded. As such, having a tractable model of uniform price auctions with complementary preferences is a crucial application of auction theory that has, to this point, resisted analysis.

In addition, we can assume that the agent valuations are affiliated with a common value component. This is crucial if we want to apply our conclusions to securities markets, goods which have an obvious and crucial common value component in terms of the stream of payouts the holders of the security receive. By including a common valuecomponent to bidder valuations, we show that information aggregation occurs and the common value component is perfectly revealed in the limit as $N \to \infty$. Therefore, our analysis can be seen as a contribution to the literature on game theoretic foundation for rational expectations equilibria. Although the question had been previously answered for single and double auction models wherein the bidders have unit demand (Reny and Perry [74]), the extension to the multi-unit demand case remained an open question due to the difficulty of analyzing the large multi-unit demand auctions.

Our approximation techniques let us use analytically tractable, incentive-free general equilibrium economies to analyze the complex, strategic models of large uniform price auction markets. Intuition suggests that a large uniform price auction would behave similarly to a general equilibrium model with the auction serving as a price discovery and information aggregation mechanism. We will use our nonatomic approximation techniques to show that under general preference distributions, the intuitive relationship between large uniform price auctions and general equilibrium models is valid. Our approximation theorems implies that the equilibrium bids in the uniform price auction are approximately turhtful, which immediately implies that the auction outcomes are approximately efficient for large N. Further, information about the common value component of the bidder valuations aggregates as $N \to \infty$.

The literature on general equilibrium economies shows that under general preference structures, equilibrium exists and is efficient (Aumann [8]) and that, when agents truthfully reveal their preferences in a nonstrategic fashion, as N approaches infinity large market equilibria converge to the equilibria of the nonatomic limit model under relatively weak technical requirements (Hildenbrand [39]). In addition to illustrating the power of our analysis framework, we provide an intuitive linkage between the nonstrategic models in the spirit of Aumann and Hildenbrand and the game-theoretic literature represented by Swinkel's study.

3.1. **Model.** We will consider a uniform price auction setting wherein the agents have quasi-linear preferences for up to $K < \infty$ homogeneous items. Agent types $\theta \in \Theta = [0, \overline{v}]^K$ represent signals that are conditionally i.i.d. given an aggregate random variable $\omega \in \Omega \subset \mathbb{R}$.[27] We will denote the conditional distribution of agent types as $\pi_0^\Theta(\omega)$ and assume $\pi_0^\Theta(\omega)$ is absolutely continuous with respect to the Lebesgue measure.[28] Agent utilities in the limit are assumed to be quasilinear in price. We require that the agent utility respond to their own type even if the state ω is known. The assumption that agent utilities are partially private allows us to avoid the Grossman-Stigliz paradox when computing the equilibrium of the nonatomic limit game.

Assumption 5. *$u(\theta, \omega, x)$ is strictly increasing and continuous in θ holding ω fixed*

The value of ω and $\pi_0^\Theta(\omega)$ is only revealed to the agents after they receive their allocations. In the nonatomic and large finite games, the agents are required to infer the value of ω from their own type and the equilibrium market price. As we will see below, the nonatomic game equilibrium is a rational expectations equilibrium (REE) in an exchange economy setting. However, since the good is divisible only with respect to aggregate demand's response to price changes, the utility functions are not differentiable with respect to outcomes and most of the prior research on REEs in this setting is not of use in analyzing our model. However, we will use analytical techniques common to the study of auctions to show that the REE of the nonatomic game is well behaved.

[27] We place an upper bound on valuations to rule out economies with infinite prices in equilibrium.

[28] The absolute continuity condition is required only to show that the large finite game is informationally (ε, ρ)-efficient.

In order to prove that the nonatomic limit game admits a unique full revelation rational expectations equilibrium, we will require that the agent bids be monotone in their type. For this we require the following

Assumption 6. *[Common Value Case]* $u(\theta, \omega, x)$ *is strictly supermodular in* (θ, ω, x)

Assumption 7. *[Private Value Case]* $u(\theta, \omega, x)$ *is strictly supermodular in* (θ, x) *and for all* ω, ω' *we have* $u(\theta, \omega, x) = u(\theta, \omega', x)$

Assumption 8. $f(\theta|\omega)$ *is strictly log supermodular in* (θ, ω)

In lemma 2, we see that these properties are enough to ensure that agent bids must be increasing in their own type. This implies that the nonatomic equilibrium will be fully revealing and hence efficient. In the case of purely private values, we require only that assumption 6 and that $\omega > \omega'$ implies $\pi_0^{\Theta}(\omega)$ weakly first order stochastically dominates $\pi_0^{\Theta}(\omega')$. Our stronger assumptions are required to deal with the joint informativeness of price and type in the common value setting.

We interpret θ_i as representing an agent's signal about the his marginal valuation for the i^{th} unit he (may) receive, which is composed partly of a common value component. We do not insist that these marginal valuations be decreasing and explicitly allow for the case of complementarities between successive units. An example of such a valuation vector in a private values setting would be (in the $K = 2$ case), $\theta = (0, 1)$, wherein the agent values only pairs of the good. By way of contrast, Swinkels [87] demands the agents to have decreasing, private marginal valuations for successive units. Our extension to the partially common value case is important for auctions of securities, items that have an essential common value component.

We define $\mathcal{M}$ as the set of monotone decreasing, upper hemicontinuous[29] demand schedules for prices within $[0, \overline{v}]$. We will use the notation $q : [0, \overline{v}] \to \{0, ..., K\}$ to denote the demand at price p given demand schedule $q \in \mathcal{M}$. Given a strategy $m : \Theta \to \mathcal{M}$, define the declared demand of an agent of type θ as $m(p; \theta)$. Note that the space of demand schedules admits a finite dimensional representation as an demand schedule is equivalent to the (finite) set of price discontinuities where an agent claims indifference between receiving $k \in \{1, .., K\}$ and $l < k$ units of the good. Given an agent's type θ, we can define the true interim demand of the agent in the nonatomic game

$$D_\theta(p) = \underset{x \in \{0,1,...,K\}}{\arg\max}\ E[u(\theta, \omega, x)|\theta, p] - p \cdot x \tag{3.1}$$

[29] Upper hemicontinuity is important in our proof to identify points of indifference. This condition is trivial if agent preferences are continuous in price.

where p represents the uniform price for the good and the expectation is taken over ex post utility given realized equilibrium market price, p, and the agent's own private valuation reflected in the type θ. The dependence of demand schedule on market price allows the agents to incorporate whatever information is contained in such prices into their utility calculation. As we will see, the price is fully revealing and hence $D_\theta(p)$ is the result of a rational expectations equilibrium calculation rather than an averaging over risk per se.

A total of M units are auctioned off in the N agent auction and assume that $\frac{M}{N} \to r \in (0, K)$ as $N \to \infty$. Given a declared demand schedule distribution $\pi^{\mathcal{M}}$, the auction makes an assignment to an agent who declared $q \in \mathcal{M}$ equal to $x(\pi^{\mathcal{M}}, q) \in \{0, ..., K\}$ at a price of $p(\pi^{\mathcal{M}}) \in [0, \overline{v}]$ for a total payment of $x(\pi^{\mathcal{M}}, q) * p(\pi^{\mathcal{M}})$. As we will be focusing on equilibria and ε-equilibria involving truthful declaration, we will not specify the tie breaking system except to note that ties will have negligible impact on efficiency or interim incentives as a realization of types in which two agents have identical valuations is a probability zero event. Let $p(\pi^{\mathcal{M}}; r)$ denote the market clearing price as a function of both the distribution of demand schedules declared to the mechanism, $\pi^{\mathcal{M}}$, and the supply of the goods, $r \in (0, K)$, available for allocation.

For the large finite auction we will use the following conditions to define the outcome

(Individual Demand) $$x(\pi^{\mathcal{M}}, q; r) \in \underset{\delta \to 0}{Lim} q(p(\pi^{\mathcal{M}}; r) + \delta)$$

(Market Clearing)
$$p(\pi^{\mathcal{M}}; r) = \sup_{p \in [0, \overline{v}]} p \text{ such that } \int q(p) * \pi^{\mathcal{M}}(dq) > \frac{M}{N} \text{ and}$$
$$\text{for all } \delta > 0, \int q(p + \delta) * \pi^{\mathcal{M}}(dq) \leq \frac{M}{N} \}$$

These conditions imply that the highest losing bid is accepted as the market clearing price, and every bidder receives the number of units demanded at this price with the exception of the highest losing bidder, for whom

$$\underset{\delta \to 0}{Lim} q(p(\pi^{\mathcal{M}}; r) + \delta) > q(p(\pi^{\mathcal{M}}; r)) \tag{3.2}$$

The highest losing bidder declared a demand schedule at which he was indifferent between two bundles at $p(\pi^{\mathcal{M}}; r)$, and his indifference is broken in favor of receiving a smaller bundle of goods.

Note that we allow the auctioneer to retain a fraction of the goods at auction in the case where the market clearing condition necessitates a strict inequality. Since we will proceed to show that the nonatomic mechanism outcomes approximate the large finite mechanism predictions, a vanishing fraction of the goods for auction will be retained by the auctioneer in the limit as the number of agents diverges to infinity. In the nonatomic case we will use a Walrasian equilibrium concept

(Individual Demand) $$x(\pi^{\mathcal{M}}, q; r) \in \underset{\delta \to 0}{Lim} q(p(\pi^{\mathcal{M}}; r) + \delta)$$

(Market Clearing) $$p(\pi^{\mathcal{M}}; r) = \sup_{p \in [0, \overline{v}]} p \text{ such that } \int q(p) * \pi^{\mathcal{M}}(dq) > r \text{ and}$$
$$\text{for all } \delta > 0, \int q(p+\delta) * \pi^{\mathcal{M}}(dq) \leq r \ \}$$

where r is taken to be the quantity of goods, treated as divisible in the nonatomic context, to be distributed amongst the agents. The individual demand condition in the Walrasian case is considered a tie breaking rule for the case where an atom of agents is indifferent between two (or more) bundles at the market clearing price $p(\pi^{\mathcal{M}}; r)$. As we will see, the equilibria of the nonatomic model are outcome equivalent to truth declaration in a Walrasian exchange economy, if the distribution of valuations π_0^{Θ} is atomless, then for almost all agents tiebreaking is not an issue.

We will propose two notions of approximate efficiency dilineated by the timing of their execution. The stronger notion of efficiency is evaluated the ex post stage.

Definition 6. *Consider an N-agent auction with supply M. The set of feasible allocations is then the set*

$$\mathcal{J} = \{x : \{1, ..., N\} \to \{1, ..., K\} \text{ such that } \sum_{i=1}^{N} x(i) \leq M\} \tag{3.3}$$

*An equilibrium outcome of the $N-$agent game, $x : \Delta(\mathcal{M}) \times \mathcal{M} \to \{0, ..., K\}$ with message strategy m^N, is **ex post ε-efficient** if*

$$\int u_N(\theta, \omega, x(\pi^{\mathcal{M}}, m^N(\theta))) * \pi^{\Theta}(d\theta) + \varepsilon \geq \max_{x^* \in \mathcal{J}} \frac{1}{N} \sum_{i=1}^{N} u_N(\theta_i, \omega, x^*(i)) \tag{3.4}$$

The ex post efficiency crtieria is very strong in the context of our model since the agents in the nonatomic game cannot condition their messages on ω except indirectly through price. As we will see below, market price alone is not sufficient to identify ω at the ex post stage of our nonatomic game, which suggests that a weaker notion of approximate efficiency will be required.

Definition 7. *Consider an N-agent auction with supply M. The set of feasible allocations is then the set*

$$\mathcal{J} = \{x : \{1, ..., N\} \to \{1, ..., K\} \textit{ such that } \sum_{i=1}^{N} x(i) \leq M\} \tag{3.5}$$

Consider the sigma algebra $\mathcal{P}$ over the space $(\Omega \times (0, K) \times \Theta)$.[30] *An equilibrium outcome of the $N-$agent game, $x : \Delta(\mathcal{M}) \times \mathcal{M} \to \{0, ..., K\}$ with message strategy m^N, is* ***informationally*** (ε, ρ)***-efficient*** *with respect to $\mathcal{P}$ if*

$$E\left[\int u_N(\theta, \omega, x(\pi^{\mathcal{M}}, m^N(\theta))) * \pi^{\Theta}(d\theta)|\mathcal{P}\right] + \varepsilon \geq \max_{x^* \in \mathcal{J}} E\left[\frac{1}{N}\sum_{i=1}^{N} u_N(\theta_i, \omega, x^*(i))|\mathcal{P}\right] \tag{3.6}$$

with probability at least $1 - \rho$

The notion of information efficiency requires that the allocation be optimal conditional on some restriction on the agents information. In addition, we demand that the agent information partition lead to near efficiency only probabilistically. This allows for the possibility that the agents inferences based on $\mathcal{P}$ could lead to suboptimal allocations with some low, but positive probability.[31] We will consider whether our common value mechanisms are efficient relative to the weaker standard of informally efficiency with respect to the information revealed by the equilibrium price function and the agents' own types. In the private value context, we will find that our uniform price auction will also be approximately ex post efficient.

We will analyze this mechanism by studying the nonatomic mechanism equilibria and then use our theorems to show that the large finite auction will yield approximately identical outcomes to the nonatomix analog. To employ our theorems, we will need to prove that the uniform price auction mechanism is continuous. When the supply r is common knowledge in the nonatomic model, the uniform price auction will have a discontinuity for an agent that declares that she is indifferent between receiving and not receiving the good at the commonly known market clearing price. To smooth this discontinuity, we will assume that there is aggregate uncertainty on the supply side that is not resolved until after the agents have submitted their valuations to the auction. We capture this uncertainty by letting r be a random variable with distribution π^r with support over some open subset of $(0, K)$.

[30] We interpret this as the space of state of the world $\omega \in \Omega$, per capita supply realizations $r \in (0, K)$, and agent type $\theta \in \Theta$.

[31] For example, in a large market it is possible that a distribution of types is realized that, in an intuitive sense, does not reflect the underlying common value. In this low probability event, it is unlikely price (or the agent inferences from price) will accurately reflect the state of the world and the good could be misallocated.

The noise in our model of the uniform price auction can be economically interpreted in a variety of ways. For example, the noise could be the result of noise traders who suffer aggregate shocks to demand. Oil lease auctions and Federal Reserve security auctions often have reserve prices that are not disclosed to bidders. Under the assumption that these reserve prices are influenced by exogenous shocks at the ex post stage, then supply is random as seen by bidders at the interim stage. The natural interpretation of the noise our continuity proof requires, as well as whether noise in aggregate demand or aggregate supply is more natural, will have to be decided on an application by application basis.

The mechanism allocation function is then redefined as $x : \Delta(\mathcal{M}) \times \mathcal{M} \rightarrow \Delta(\{0, ..., K\})$ and the price function becomes $p : \Delta(\mathcal{M}) \rightarrow \Delta([0, \overline{v}])$. Given a realization of $r \in (0, K)$ drawn after the agents submit their valuations, the allocation and price functions are determined as per the Individual Demand and Market Clearing conditions. To agents submitting demand schedules at the interim stage, the market clearing process based on the random supply variable r is perceived as a lottery over allocations and prices. We will use the weak-* topology over the spaces $\Delta(\{0, ..., K\})$ and $\Delta([0, \overline{v}])$. We require that our agent utilities be continuous in the weak-* topology over lotteries in $\Delta(\{0, ..., K\})$ and $\Delta([0, \overline{v}])$.

As a preliminary step in the analysis, we note that the nonatomic model assumes supply, r, is a continuous variable whereas it is by definition discrete in each of the finite models. In order to handle this changing feasibility constraint as N approaches infinity, we define a family of finite mechanisms $\mathcal{O} = \{(x_N, p_N) : \Delta(\mathcal{M}) \times \mathcal{M} \rightarrow \Delta(\{0, ..., K\}) \times \Delta([0, \overline{v}])\}_{N=1}^{\infty}$ where outcomes for mechanism (x_N, p_N) are defined by the relevant finite market clearing condition with random supply M_N. To employ our theorems, we must show pointwise convergence of the sequence $\{(x_N, p_N)\}_{N=1}^{\infty}$ to the nonatomic outcome function. Therefore, we must assume that the random process $\frac{M_N}{N}$ converges to r. In this context, it is sufficient that $\frac{M_N}{N}$ converge to r in the Kolmogorov metric, although it is trivial to choose M_N so that convergence is almost sure in the strong topology. For example, we could choose $M_N = round[N * r]$ where $round[\circ]$ generates the nearest integer to $N * r$.

Our first task is to show that the agent preferences generate aggregate behavior in the nonatomic model that is well behaved, in the sense that mechanism outcomes are well defined by the Individual Demand and Market Clearing conditions. However, the nonatomic analog is simply a general equilibrium exchange economy consisting of an auctioneer and the measure 1 continuum of bidders. There are 2 goods in the economy, the good for auction and money serving as a numeraire good. From Aumann [7] we can be assured that equilibrium is well defined in this game.

3.2. **Applying our Framework.** In order to use our theorem, we need to find the equilibrium of the nonatomic version of the mechanism. Given that price cannot be affected by the declarations of individual agents in the nonatomic mechanism and since agents receive an optimal allocation given their declared demand schedules and the market clearing price, it is obvious that truthfully announcing demand schedules is an optimal message for the agents. Any other message that has an effect on the outcome will yield a suboptimal allocation for the deviating agent at the equilibrium price, which the agents treat as exogenous in the nonatomic model. In all equilibria agents declare a demand schedule that is outcome equivalent to a truthful declaration, from Aumann [10] we know that the equilibrium of the nonatomic mechanism exists and is efficient for each realization of r even in the case where agents have nonconvex preferences over the quantity of the good they are allocated.[32]

To show that the nonatomic equilibrium outcomes identified above approximate the equilibria of the large finite auctions, we need to show that the mechanism is continuous in the relevant topologies. As a first step, we will assume that the agents only employ strategies that are rationalizable. This implies that if the agent's true interim demand is

$$D_\theta(p) = \operatorname*{arg\,max}_{x \in \{0,1,\ldots,K\}} E[u(\theta,\omega,x)|\theta,p] - p \cdot x \tag{3.7}$$

then the agent will declare a demand schedule that is weakly smaller than $D_\theta(p)$. We refer to this as our **Rationalizability Assumption**. This implies that each agent gets weakly fewer units than she desires at any price p given her true interim utility.

Lemma 1. *If the Rationalizability Assumption holds, then*

- $p(\cdot)$ *is uniformly continuous and* $\{x(\cdot,q)\}_{q\in\mathcal{M}}$ *is uniformly equicontinuous.*
- $$E[u(\theta,\omega,x(\pi^{\mathcal{M}},m))|\theta,p] - p(\pi^{\mathcal{M}},m) \cdot x \tag{3.8}$$
  is upper semicontinuous in $\Delta(\mathcal{M}) \times \mathcal{M}$ *where* $\Delta(\mathcal{M})$ *is endowed with the weak-* topology and* $\mathcal{M}$ *with the Euclidean norm.*

To gain some intuition for a this theorem, consider a distribution of demand declarations $\pi^{\mathcal{M}}$. The market clearing price is set by agents on the margin who claim to be indifferent between either receiving k or $l > k$ units of the good at the market clearing price. In the proof of lemma 1, we show that the price determination conventions of the nonatomic

[32] The agents have a strict incentive to declare truthfully for all prices in the support of $p(\pi_0)$. The incentives to declare demand truthfully for prices never realized in equilibrium are not strict and any outcome equivalent declaration is an equilibrium strategy.

uniform price auction imply that if a small mass of bidders at the margin moves, the price will shift only a small amount. This implies continuity of the interim price function in the weak-* topology. However, to establish continuity of the allocation function, we require that the interim price allocation be continuous in the stronger Kolmogorov topology. This requires noting that if the marginal bidders are part of an atom of bidders at demand declaration q, for small enough changes in the distribution of declared demands (under the Kolmogorov topology), an atom of marginal bidders will remain at q and the market clearing price will remain unchanged. The stationarity of atoms in the price distribution strengthens continuity under the weak-* topology to continuity under the Kolmogorov metric.

The proof of the continuity of the allocation functions, $\{x(q, \pi^{\mathcal{M}})\}_{q \in \mathcal{M}}$, relies on the fact that an agent's stochastic allocation is determined by the interaction of his declared demand schedule and realized market clearing price falls. From a simple revealed preference argument, we can show that demand for any type of agent is weakly decreasing in price, which implies the agent's demand schedule can be defined as a (finite) set of upper contour sets in the space $[0, \overline{v}]$ of prices. Because the interim market clearing price, $p(\pi^{\mathcal{M}})$, is continuous in the Kolmogorov topology on the range, the probability that the interim price lies in one of these upper contours is continuous in $\pi^{\mathcal{M}}$. Continuity of the allocation function in the follows immediately.

Lemma 2. *The unique equilibrium of the nonatomic game is a fully revealing rational expectations equilibrium in which agents use bidding strategies increasing in their own type. The ex post price, $p(\omega, r)$, and supply, r, are sufficient to fully reveal ω.*

Given we have established that our mechanism is continuous and presumed that our utility function is continuous as well, it remains to derive the equilibrium behavior of our nonatomic limit economy. In the nonatomic limit game, the price function $p(\pi_o^{\mathcal{M}})$ is exogenous to any agent's individual decision. Therefore, agents have no incentive to lie and any potential equilibrium will be truthful over the range of price realized in equilibrium. Therefore, agents will declare a demand schedule equal to

$$D_\theta(p) = \underset{x \in \{0,1,\ldots,K\}}{\arg\max}\ E[u(x, \omega, \theta)|\theta, p] - p \cdot x \tag{3.9}$$

where the expectation takes into account the common knowledge of the equilibrium strategies of the agents in the nonatomic game. Since we have assumed that $u(\theta, \pi^{\Theta}, x)$ is strictly increasing in θ and supermodular and the conditional distribution of types is log supermodular, the demand schedules of the agents in the limit game will be strictly increasing in θ. We show that this implies that the ex post price function $p(\omega, r)$ is strictly monotone in (ω, r). Therefore $p = p(\omega, r)$ and the value r are sufficient to discover ω.

We can show that the nonatomic model is informationally efficient with respect to the equilibrium price function $p(\omega, r)$. Intuitively, this is because the agents incorporate all information available with respect to this sigma algebra into the computation of their optimal demand schedule declaration. In the case of private values, as their is no value in conditioning on the equilibrium price, we find that the nonatomic uniform price auction is ex post efficient.

Lemma 3. *In the Private Values Case, the nonatomic equilibrium is ex post efficient. In the Common Values Case, let the sigma algebra* $(\Omega \times (0, K), \mathcal{P})$ *be the coarest such that* $p(\omega, r)$ *is measurable. Finally, let* $\mathcal{P}^*$ *be the coarsest sigma algebra containing* $\mathcal{P} \times \mathcal{B}(\Theta)$. *Then the allocation is informationally efficient with respect to* $\mathcal{P}^*$.

Given the nonatomic limit game equilibrium, we can then claim from our theorem 11 the following result on the asymptotic behavior of our large uniform price auctions:

Theorem 7. *For any* $\varepsilon > 0$, $\rho \in [0, 1)$ *we can choose* N^* *such that for* $N > N^*$ *the equilibrium of the* N *agent auction has the following properties*

- *Ex post* ε*-efficient in the Private Values Case*
- *Informationally* (ε, ρ)*-efficient with respect to* $\mathcal{P}^*$, *where* $(\Omega \times (0, K), \mathcal{P})$ *be the coarest such that* $p(\omega, r)$ *is measurable and* $\mathcal{P}^*$ *is the coarsest sigma algebra containing* $\mathcal{P} \times \mathcal{B}(\Theta)$
- *The* N*-agent equilibrium price distribution,* $p_N(\omega, r) : \Omega \times (0, K) \to \Delta([0, \overline{v}])$, *converges to* $p(\omega, r)$ *almost surely.*

Analyses of the equilibria of large finite auctions are complicated by the strategic interactions of the bidders, which in the case of the uniform price auction involves bidders withholding demand to lower the market clearing price. Our analysis was simplified since the nonatomic form of the auction reduces to a stochastic Walrasian economy that can be analyzed without the preference restrictions required to employ game theoretic techniques. Once the mechanism was shown to be continuous, our main theorems let us employ the nonatomic game as an approximation to the large finite game. Therefore, the efficiency of the Walrasian equilibria implies ε-efficiency of the large finite auction. In addition to the novelty of the result, this analysis demonstrates the simplicity with which mechanisms can be analyzed using our techniques.

Our approximation theorems, most notably Theorem 4, prove that the equilibrium correspondence is upper hemicontinuous in N. If the set of equilibria is empty in the large finite game, then our approximation theorems are of little use. However, if we finitely discretize the space of bids, we can use Milgrom and Weber [60] to show existence of a mixed strategy in the large finite games. Combined with Theorem 7, we see these mixed strategies approach strategies that are approximately outcome equivalent to an efficient truthful strategy for a sufficiently fine gridding.[33] Efficiency in turn implies the monotonicity of McAdams [56] as $N \to \infty$.

3.3. **Extension.**

3.3.1. *Private Values.* Consider the private values formulation. In this setting there is no question of extracting information from prices as in a rational expectations equilibrium. Therefore, the nonatomic limit game is a private values general equilibrium exchange economy. The only condition we require on the agent valuation functions is that they be weak-* continuous in the random allocation-price that the agent receives. We could extend our analysis in the private value case to:

- Expected utility preferences with or without a common prior[34]
- Risk averse preferences
- Agent valuations are drawn from a (finite) set of assymetric valuation distributions
- Agents are loss averse with any reference point

For each of these preference relations, truthful revelation of demand remains an equilibrium of the private values nonatomic limit game. Therefore, equilibria of the large finite game will be approximately truthful and approximately efficient. We conjecture that truthful revelation would remain an equilibrium of the partially private values nonatomic limit game, but properties such as approximate efficiency and information aggregation may not hold.[35]

[33] In this context, the degree of approximation is increasing in both the fineness of the gridding and the number of agents.

[34] We show that truthful declaration is an equilibrium of the limit game, but agents only have strict preferences to declare truthfully for prices realized in equilibrium. For the entire set of equilibria of the nonatomic limit game to be outcome equivalent we require a restriction on the admissable agent priors. Specifically, agents at the interim stage must assign positive probability to the event that the ex post price lies in any interval of prices that occur in a truthful equilibrium.

[35] For example, if the agents hold heterogeneous priors over the distribution of types in the economy, it could be the case that different agents draw different inferences from the same ex post price realization. It is not clear whether information will aggregate or efficient outcomes will be realized in the uncommon prior setting. This is an interesting question for future reesearch.

3.3.2. *Multiple Types of Goods.* Finally, our proof techniques could be extended to an auction for multiple types of goods, but the problem in this case becomes defining a price mechanism. In the single good case, the market clearing price is defined to fall somewhere between the highest losing and lowest winning bid. Any choice within this range suffices to reproduce our results and is unique in the allocation outcome. However, the most natural set of market clearing conditions for an economy with L types of good with M_l units of good l to be allocated are

(Individual Demand) $$x(\pi^{\mathcal{M}}, q; r) \in q(p(\pi^{\mathcal{M}}; r))$$

(Market Clearing)

$$p(\pi^{\mathcal{M}}; r) \in \Lambda^P = \{p : \int q(p) * \pi^{\mathcal{M}}(dq) \leq \frac{M}{N} \text{ and}$$
$$\text{for all } \delta \in \mathbb{R}_+ \backslash \{0\},\ \int q(p-\delta) * \pi^{\mathcal{M}}(dq) \nleq \frac{M}{N}\ \}$$

where the price is selected from Λ^P. Even if we assume that the market clearing condition is supplemented with a selection rule such as

(Revenue Maximization) $$p(\pi^{\mathcal{M}}; r) = \underset{p \in \Lambda^P}{\arg\max} p \cdot \int q(p) * \pi^{\mathcal{M}}(dq)$$

it is not clear in our context that a unique price would result.

An alternative solution would be to consider L large uniform price auctions, one auction for each of the types of goods. Once agent preferences are revealed, the auctions are resolved in random order. This would be equivalent to the following procedure:

(1) Choose a permutation, $\sigma : \{1, .., L\} \to \{1, ..., L\}$
(2) Let $\Lambda_0^P(\sigma) = \Lambda^P(\sigma)$
(3) For $i \in \{1, .., L\}$, let $\Lambda_i^P(\sigma) = \{p : p \in \Lambda_{i-1}^P(\sigma)$ and for all $p' \in \Lambda_{i-1}^P(\sigma)$, $p_{\sigma(i)} \geq p'_{\sigma(i)}\}$

This procedure obviously results in a selection of a price that is continuous in the Hausdorff metric on Λ^P. It remains to show that Λ^P is continuous in this metric if $\int q(p) * \pi^{\mathcal{M}}(dq)$ is continuous in the Kolmogorov metric over $\pi^{\mathcal{M}}$, which is trivial to show through a backward induction argument given the proofs we have already completed.[36]

4. Dynamic Games and Econometrics

The principle theorems of this study are developed for static mechanisms. However, there are useful cases in which the results can be extended to dynamic games and mechanisms. Dynamic competitive economy models feature a measure 1 continuum of agents who respond optimally to their correct expectations about market aggregates in the present and all future periods. These models presume that market aggregates, such as market

[36] The only subtlety is to note that continuation values for the $L - 1^{st}$ auction are functions of the (continuous) outcome of auction L. Therefore, an argument as presented here applied to the $L - 1$ auction outcome function, combined with continuity of the continuation values, implies continuity of the $L - 1^{st}$ auction. Induction implies continuity of all L auctions.

price, are exogenous to a single agent's action but are consistent with the aggregate distribution of actions taken by all agents. An alternative modeling technique uses large stochastic games and game theoretic equilibrium concepts in which each agent is presumed to respond optimally given the agent's hierarchy of beliefs. A generic agent i's hierarchy contains beliefs about the actions of other agent's, the beliefs about other agents' beliefs about the action of agent i, beliefs about the other agents' beliefs about agent i's beliefs about the beliefs of other agents' beliefs about agent i's action, ad infinitum. Typically stochastic games become computationally intractable to solve or estimate with more than a handful of agents. On the other hand, it is usually uncertain how well the continuum of agents in a dynamic competitive economy model capture the behavior of a large, but finite, number of agents in the real economy.

The focus of our work is to provide tools for practitioners interested in either computationally solving or econometrically estimating large stochastic games. The closest work to ours is a series of papers by Benkard et al. ([15], [16],[17]), whose analysis is restricted to a model of industry competition formulated first by Ericson and Pakes [29]. Our work provides a general framework for approximating large stochastic games with dynamic competitive models and, importantly, the conditions under which our approximation techniques are valid are based on the primitives of the model. The analyst need not conduct any game theoretic analysis to verify that our techniques are applicable.

The possibility of this extension may surprise readers familiar with Green [35] and Sabourian [81], both of which required stronger topological notions of continuity than we have employed to generate limits relating large finite dynamic games with the nonatomic analogs. In lieu of these more complex topological notions, we will be required to place restrictions on the equilibrium strategies of the agents. While the restriction of endogenous outcomes is undesirable, the restricted strategy space is natural in a variety of settings of interest to applied theorists and econometricians.

We will consider an N agent repeated game with the goal of analyzing the equilibria as $N \to \infty$. As in the sections above, we will employ a nonatomic game played by a measure 1 continuum of agents as an approximation of a game played by a large finite number of agents. Our goal will be to establish sufficient conditions for the equilibria of the nonatomic limit game to serve as approximations of the equilibria of games with a large finite number of agents. Further, given the sufficient conditions we outline, we will argue that the equilibria of the large finite games are in practice more difficult to compute or estimate than equilibria of the analogous nonatomic limit games.

Proofs for the theorems of this section are omitted for length, but are available from the author upon request.

4.1. **Model.** Agent actions are drawn from a finite dimensional Polish metric space $(\mathcal{A}, d_{\mathcal{A}})$. Since we will allow mixed strategies, we will also require the space of lotteries over $\Delta(\mathcal{A})$ endowed with the weak-* topology.[37] The agent type space is a set $\Theta \subset \mathbb{R}^d$. The model admits an exogenous aggregate shock term we denote $\varphi \in \Psi \subset \mathbb{R}^d$, where we treat Ψ as a subset of the d-dimensional Euclidean metric space. We will assume that the aggregate shock term is Markovian with transition probability function $Y : \Psi \times \mathcal{B}(\Psi) \to [0, 1]$, where $\mathcal{B}(\Psi)$ is the standard Borel sets over Ψ.

Our state of the economy is comprised of two components. The first component is the distribution of agent types in the economy, $\pi^\Theta \in \Delta(\Theta)$, endowed with the weak-* topology.[38] The second component is the current value of the aggregate shock term, $\varphi \in \Psi$. The state space is then the product space $\mathrm{S}{=}\Delta(\Theta) \times \Psi$ with a generic state denoted $\mathrm{s} = (\pi^\Theta, \varphi) \in \mathrm{S}$. When we are dealing with N-agent economies, the first component of the state will be restricted to those measures that can be generated by the empirical distribution of the N-agent types, and we denote the space of such $N-$agent compatible empirical measures as $\Delta^N(\Theta)$. We denote this restricted state space as $\mathrm{S}_N \subset \mathrm{S}$ and endow S_N with the relative topology inherited from S. Where required, we will treat S as a metric space $(\mathrm{S}, d_{\mathrm{S}})$ where

$$d_{\mathrm{S}}(\mathrm{s} = (\pi^\Theta, \varphi), \widetilde{\mathrm{s}} = (\widetilde{\pi}^\Theta, \widetilde{\varphi})) = d_{LP}^\Theta(\pi^\Theta, \widetilde{\pi}^\Theta) + \|\widetilde{\varphi} - \varphi\| \tag{4.1}$$

where $\|\circ\|$ refers to the Euclidean norm on Ψ. In equilibrium, this is equivalent to a model that allows payoffs to be affected by the aggregate actions of the agents in the market (ex: price distributions from competitors) or state variables that are a function of either agent types or actions (ex: aggregate production or investment).

The utility of each agent in each period of the N-agent game is generated by the felicity function $w_N : \Theta \times \mathrm{S} \times \mathcal{A} \to \mathbb{R}$ where $w_N(\theta, \mathrm{s}, a)$ is the utility payoff for an agent taking action a given his own type θ and state of the economy s. In taking the limit as $N \to \infty$, we will employ a family of felicity functions $\{w_N : \Theta \times \mathrm{S} \times \mathcal{A} \to \mathbb{R}\}_{N=1}^\infty$. The restriction to this form of utility is significant in two regards. First, the agent utility is affected only by the anonymous state of the economy. If two agents in the economy swap types, the utility of the other agents will not be affected. Second, utility of a given agent in

[37] We let $d_{LP}^{\mathcal{A}} : \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \to \mathbb{R}_+$ denote the Levy-Prokhorov metric that metrizes the weak-* topology over $\Delta(\mathcal{A})$.

[38] We let $d_{LP}^{\Theta} : \Delta(\Theta) \times \Delta(\Theta) \to \mathbb{R}_+$ denote the Levy-Prokhorov metric that metrizes the weak-* topology over $\Delta(\Theta)$.

the present period is not affected by the actions taken by the other agents in the present period. However, as we see below, other agents' actions in the present period affect the state of the economy realized in the future, so the actions of others will influence an agent's value function in the present period through the continuation values. The felicity function of the nonatomic limit game will be denoted $w : \Theta \times \mathrm{S} \times \mathcal{A} \to \mathbb{R}$. Intertemporal utility at time t is then

$$U_N(\mathbf{a}) = (1-\delta) * E_t \left[\sum_{\tau=0}^{\infty} \delta^{\tau} w_N(\theta^i_{t+\tau}, \mathrm{s}_{t+\tau}, a_{t+\tau}) \right] \tag{4.2}$$

$$U(\mathbf{a}) = (1-\delta) * E_t \left[\sum_{\tau=0}^{\infty} \delta^{\tau} w(\theta^i_{t+\tau}, \mathrm{s}_{t+\tau}, a_{t+\tau}) \right] \tag{4.3}$$

where $\delta \in (0,1)$ is the time discount factor, $\mathbf{a} = (a_0, a_1, ...)$ with a_t being the vector of per-period actions of the agents at time t, θ^i_t is agent i's type at time t, and π^{Θ}_t is the distribution of types in the economy at time t.

The per-period payoffs in the limit game ($N \to \infty$) are determined by the function $w : \Theta \times \mathrm{S} \times \mathcal{A} \to R$, which assumes that all payoff relevant effects are captured by market conditions $\mathrm{s} \in \mathrm{S}$, the agent's type $\theta \in \Theta$, and the agent's reaction to these conditions $a \in \mathcal{A}$. The following assumption implies that the preferences of the agents in the N-agent game approach the preferences of the agent in the limit game as $N \to \infty$.

Assumption 9. $w(\theta, \mathrm{s}, a) = \underset{N\to\infty}{Lim}\, w_N(\theta, \mathrm{s}, a)$ *uniformly for all* $(\theta, \pi^{\Theta}, a) \in \Theta \times \mathrm{S} \times \mathcal{A}$

We then assume[39]

Assumption 10. $\{w(\theta, \cdot, a) : \mathrm{S} \to \mathbb{R}\}_{\Theta \times \mathcal{A}}$ *is uniformly equicontinuous.*

Assumption 11. $w : \Theta \times \mathrm{S} \times \mathcal{A} \to \mathbb{R}$ *is bounded.*

We assume an agent's type evolves according to the probability transition function $T : \Theta \times \mathcal{B}(\Theta) \times \mathrm{S} \times \mathcal{A} \to [0,1]$ where $\mathcal{B}(\Theta)$ refers to the standard Borel sets on Θ endowed with the Hausdorff metric. For a present state of the economy $\mathrm{s} = (\pi^{\Theta}, \varphi) \in \mathrm{S}$, agent action $a \in \mathcal{A}$, current agent type $\theta \in \Theta$, and target set $U \in \mathcal{B}(\Theta)$, the probability that an agent of type θ_t at time t has a type $\theta_{t+1} \in U$ next period is $T(\theta_t, U; \mathrm{s}, a)$. Conditional on the state of the economy, s, agent type transitions are stochastically independent. Where confusion will not result, we will treat $T(\theta, \circ; \mathrm{s}, a)$ as a measure over Θ and endow the space of such measures with the weak-* topology.

[39]Note that uniform continuity of $w : \Theta \times \mathrm{S} \times \mathcal{A} \to \mathbb{R}$ implies our continuity assumption.

The type transition function allows the effect of an action, $a \in \mathcal{A}$, to depend both on the agent's current type and the state of the economy. For example, the dependence on present type can be used to reflect higher capital depreciation level when capital stocks are high. The dependence of the transition function on the state of the economy can be used to reflect influence of the aggregate state on the effectiveness of actions taken by a firm. For example, in a model of R&D races, if the agent's action reflects investment in a research project and the type reflects a stock of intellectual capital, then economies where firms are aggressively pursuing research projects may make it less likely that a particular firm's research project wins the race to a novel discovery. This would reduce the effectiveness of a given level of research expenditure when research level is high on average.

In some portions of this study we will restrict our analysis to a subset of the set of possible type evolution operators that are independent of the state of the economy.

Definition 8. *The type evolution operator* $T(\theta, \circ; \mathrm{s}, a)$ *is* ***state independent*** *if for all* $\mathrm{s}, \mathrm{s}' \in \mathrm{S}$*, we have* $T(\theta, \circ; \mathrm{s}, a) = T(\theta, \circ; \mathrm{s}', a)$

4.2. **Equilibrium Definitions.** The per-period payoffs in the limit game ($N \to \infty$) are determined by the function $w : \Theta \times \mathrm{S} \times \mathcal{A} \to R$, which assumes that all payoff relevant effects are captured by market conditions $\mathrm{s} \in \mathrm{S}$, the agent's type $\theta \in \Theta$, and the agent's reaction to these conditions $a \in \mathcal{A}$. We will let the strategy space, denoted Σ, consist of all measurable maps from $\Theta \times \mathrm{S}$ into $\Delta(\mathcal{A})$ that are continuous in S. Let Σ^C denote the elements of

Assumption 12. *The agents are restricted to choosing Markovian strategies* $\sigma : \Theta \times \mathrm{S} \to \Delta(\mathcal{A})$ *that are continuous in* S.

Agent i's discounted expected utility in the N-agent game, assuming the use of symmetric strategies by the agents, can then be written in value function form

$$V_N(\theta_t^i, \mathrm{s}_t | \sigma) = (1 - \delta) * \left[E_t[w_N(\theta_t^i, \mathrm{s}_t, \sigma(\theta_t^i, \mathrm{s}_t))] + \delta E_t \left[V_N(\theta_{t+1}^i, \mathrm{s}_{t+1} | \sigma) | \mathrm{s}_t \right] \right] \tag{4.4}$$

The term $E_t[w(\theta_t^i, \mathrm{s}_t, \sigma(\theta_t^i, \mathrm{s}_t))]$ reflects an expectation over the mixed strategy σ, whereas $E_t \left[V_N(\theta_{t+1}^i, \mathrm{s}_{t+1} | \sigma) | \mathrm{s}_t \right]$ is an expectation taken over continuation values with respect to next period's type and next period's state of the economy. Note that the above formulation employs the symmetry restriction at two points. First, we are able to drop the notational complexity of separately tracking the strategies of the agents. Second, and more importantly, the value function can be written semi-anonymously since the distribution of types and the symmetric strategy are sufficient to determine the distribution of future types and actions. In order to consider optimality conditions, we will let $V_N(\theta_t^i, \mathrm{s}_t | \sigma_i', \sigma_{-i})$

denote the utility of agent i in the N-agent game when he follows strategy σ_i' and all other agents follow strategy σ.

Definition 9. *A symmetric $(\varepsilon,\rho)-$Markov Perfect Equilibrium ($\varepsilon-$MPE) strategy and state pair $(\sigma^{MPE},\mathrm{s}_0)\in\Sigma\times\mathrm{S}$ of the N-agent game such that for all agents $i\in\{1,...,N\}$ we have*

$$\sup_{\sigma'\in\mathcal{A}} V_N(\theta_t^i,\mathrm{s}_t|\sigma_i',\sigma_{-i}^{MPE})\leq V_N(\theta_t^i,\mathrm{s}_t|\sigma^{MPE})+\varepsilon \text{ for all } \theta\in\Theta,\mathrm{s}_t\in\mathrm{S} \tag{4.5}$$

for at least a measure $1-\rho$ of states s_t of the economy realized along the path of play commencing at state s_0.

The equilibrium is not exact since we only require that the strategies provide a payoff within ε of the feasible optima. Second, we require that the strategy only be approximately optimal a fraction $1-\rho$ of the periods along the path of play. From the full support of T, the restriction to states realized along the path of play is vacuous since all state of the economy are, potentially, on the future path of equilibrium play.[40]

In the nonatomic dynamic game, a continuum of agents participate in the game in each period. The value function formula for the nonatomic limit game when all agents play the symmetric strategy σ is then

$$V_\infty(\theta_t^i,\mathrm{s}_t|\sigma)=(1-\delta)*\left[E_t[w(\theta_t^i,\mathrm{s}_t,\sigma(\theta_t^i,\mathrm{s}_t))]+\delta E_t\left[V(\theta_{t+1}^i,\mathrm{s}_{t+1}|\sigma)|\mathrm{s}_t\right]\right] \tag{4.6}$$

The term $E_t[w(\theta_t^i,\mathrm{s}_t,\sigma(\theta_t^i,\mathrm{s}_t))]$ reflects an expectation over the mixed strategy σ, whereas $E_t\left[V_N(\theta_{t+1}^i,\mathrm{s}_{t+1}|\sigma)|\mathrm{s}_t\right]$ is an expectation taken over continuation values with respect to next period's type and next period's aggregate shock. In order to state our approximation theorems we need to define the nonatomic equivalent of an MPE. We will define a general notion of equilibrium in a dynamic continuum economy with aggregate uncertainty.

Definition 10. *A symmetric $\varepsilon-$Dynamic Competitive Equilibrium (ε-DCE) consists of a strategy $\sigma^{DCE}:\Theta\times\mathrm{S}\to\Delta(\mathcal{A})$ such that for any $s_t=(\pi_t^\Theta,\varphi_t)\in\mathrm{S}$*

(1) *For all $U\in\mathcal{B}(\Theta)$*

$$\pi_{t+1}^\Theta(U)=\int_{\mathcal{A}\times\Theta}T(\theta,U;\mathrm{s}_t,\sigma^{DCE}(\theta,\mathrm{s}_t)[da])*\pi_t^\Theta(d\theta)$$

(2) *For all $\theta\in\Theta$ and $a\in\mathcal{A}$ we have*

$$V_\infty(\theta_t^i,\mathrm{s}_t|\sigma^{DCE})+\varepsilon\geq E_t[w(\theta_t,(\pi_\infty^\Theta,\varphi),a)]+\delta E_t\left[V_\infty(\theta_{t+1}^i,\mathrm{s}_{t+1}|a_i,\sigma_{-i}^{DCE})|\mathrm{s}_t\right]$$

[40]The usual notion of Markov Perfect Equilibrium requires optimality at every state, whereas our $(0,0)-$MPE notion allows very suboptimal bevaior at a measure 0 set of states.

Note that given the state of the economy at time t, the distribution of types at time $t+1$ is common knowledge. The only uncertainty to be resolved at time $t+1$ is with respect to the aggregate shock and each agents' own type at $t+1$. The state at $t+1$ is determined by the strategy of the measure 1 of other agents, σ_{-i}^{DCE}, and the exogenous distribution of aggregate shocks. The agent's own deviation can only affect his present period payoff and his future distribution of types. Also note that the ε term is used to denote the degree of approximation of the agent's best response to the present state of the economy and future uncertainty. The optimality criteria is applied across all states, even those that are not reachable along the equilibrium path.

A common restriction of the competitive dynamic equilibrium notion defined above is that of a stationary equilibria, a nonatomic equilibrium concept wherein the economy is assumed to have no aggregate uncertainty and the state of the economy is reduced to a distribution of agent types that remains constant over time. We will assume a measure one mass of agents whose types at $t=0$ are generated according to the type distribution π_∞^Θ. Equilibrium will be defined as follows:

Definition 11. *Assume $\Psi = \{\varphi\}$ is a singleton. A symmetric $\varepsilon-$Stationary Equilibrium (ε-SE) consists of a strategy $\sigma^{SE} : \Theta \to \Delta(\mathcal{A})$ and type distribution $\pi_\infty^\Theta \in \Delta(\Theta)$ such that*

(1) *For all $U \in \mathcal{B}(\Theta)$,*

$$\pi^\Theta(U) = \int_{\mathcal{A}\times\Theta} T(\theta, U; (\pi^\Theta, \varphi), \sigma(\theta)[da]) * \pi^\Theta(d\theta)$$

(2) *For all $\theta \in \Theta$ and $a \in \mathcal{A}$ we have*

$$w(\theta_{t+\tau}, (\pi_\infty^\Theta, \varphi), \sigma(\theta_{t+\tau})) + \varepsilon \geq w(\theta_t, (\pi_\infty^\Theta, \varphi), a)$$

Conditions (1) implies that endogenous quantities are stationary given the equilibrium strategies.[41] Condition (2) implies that for each type, the action dictated by the strategy is optimal given the stationary, endogenous quantities. Note also that our optimality criteria is only applied at the stationary state $(\pi_\infty^\Theta, \varphi)$ and not off the equilibrium path. For generic games it will not be the case that the approximate optimality condition will hold generically off the equilibrium path. The usefulness of this equilibrium concept lies in the fact it is finite dimensional and hence computationally tractable rather than it's power as an equilibrium selection tool. For existence results on exact DCE or SE in continuum games, please refer to Bergin and Bernhardt [20].

[41] An alternative definition of an ε-SE weaken condition 1 to:

(1) For all $U \subset \Theta$, $|\pi^\Theta(U) - \int_{\mathcal{A}\times\Theta} T(\theta, U; (\pi^\Theta, \varphi), \sigma(\theta)[da]) * \pi^\Theta(d\theta)| < \varepsilon$

The crucial conceptual difference between the large finite game and the continuum analog is that the in an MPE of the large finite game, the state of the economy is allowed to change in response to agent actions. In the continuum game, the state of the economy is exogenous to any agent's action. We define the state transition function of the continuum game without aggregate uncertainty as $P^C(\pi_t^\Theta) = \pi_{t+1}^\Theta$ where for any $U \in \mathcal{B}(\Theta)$ [42]

$$\pi_{t+1}^\Theta(U) = \int_{\mathcal{A}\times\Theta} T(\theta, U; \mathrm{s}_t, a) * \sigma(\theta)[da] * \pi_t^\Theta(d\theta) \tag{4.7}$$

When there is aggregate uncertainty in the model, the state transition probability function of the continuum has the form

$$\text{For } V \subset \mathrm{S},\ P_\infty^A(\mathrm{s}_t = (\pi_t^\Theta, \varphi_t), V) = \int_V 1\{P^C(\pi_t^\Theta) = \pi_{t+1}^\Theta\} * Y(\varphi_{t+1}|\varphi_t) * d\mathrm{s}_{t+1} \tag{4.8}$$

where we denote $\mathrm{s}_{t+1} = (\pi_{t+1}^\Theta, \varphi_{t+1}) \in \Delta(\Theta) \times \Psi = \mathrm{S}$.

4.3. **Dynamics.** We will focus our analysis on the properties of the dynamics of the state of the economy induced by the combination of the symmetric agent strategies, $\sigma \in \Sigma$, and the type evolution function, T. Where required we will explicitly refer to the state of the economy as a random variable s:$\Omega \times \mathbb{N} \to \mathrm{S}$. In this formulation, Ω refers to an underlying probability space that defines the aggregate and idiosyncratic uncertainty of each of a countable infinity of agents for an infinite, discrete time horizon.[43] The continuum model state of the economy can be described the random variable s^C:$\Omega \times \mathbb{N} \to \mathrm{S}$.

Consider an $N-$agent economy with a nonanonymous state space $\Theta^N \times \Psi$.[44] Let $e : \Theta^N \to \Delta^N(\Theta)$ denote the Lebesgue measurable map that converts $\overrightarrow{\theta} = (\theta^1, ..., \theta^N) \in \Theta^N$ into empirical distributions $e(\overrightarrow{\theta}) = \frac{1}{N}\sum_{i=1}^N 1\{\theta^i \in A\}$.[45] Note that e is continuous. We adopt the convention that for any $U \in \mathcal{B}(\Theta^N)$, $e(U) = \cup_{\overrightarrow{\theta} \in U} e(\overrightarrow{\theta})$. Let the correspondence $e^{-1} : \Delta^N(\Theta) \rightrightarrows \Theta^N$ denote the inverse of $e : \Theta^N \to \Delta^N(\Theta)$ with the convention that for V $\subset \mathrm{S}_N$

$$e^{-1}(V) = \cup_{\pi^\Theta \in V} e^{-1}(\pi^\Theta) \tag{4.9}$$

42The mapping P^C is not a transition probability function. Our change in notation preserves the differentiation between the deterministic and stochastic components of our analysis.

43The state space for any model with $N < \infty$ is embedded in the space Ω Note that we require our state space to define the idiosyncratic shocks for a countable infinity of agents so that we can consider limits of the form $N \to \infty$ without redefining our probability space.

44The space Θ^N is nonanonymous in that types are assigned to each agent rather than aggregated into a state of the economy $\pi^\Theta \in \Delta(\Theta)$.

45We would like to define a family of functions $\{e_N : \Theta^N \to \Delta(\Theta)\}_{N=1}^\infty$. However we will simply refer to a single function e and leave it to the reader to discern the proper interpretation. This should not yield confusion within the context of our proofs.

Consider a symmetric strategy $\sigma : \Theta \times \mathrm{S} \to \Delta(\mathcal{A})$. For any $U \subset \Theta^N$, the operator relating the distribution of types at time t to the to the distribution at $t+1$ can be defined as a Markov transition probability function, $P_N : \Theta^N \times \Psi \times \mathcal{B}(\Theta^N) \to [0,1]$, that describes the probability of a state $\overrightarrow{\theta} \in \Theta^N$ transitioning into an open set $U \in \mathcal{B}(\Theta^N)$ in the next period. For a rectangular set $U = U_1 \times ... \times U_N$ where $U_i \subset \Theta$, let

$$P_N(\overrightarrow{\theta}_t, \varphi_t, U) = \prod_{i=1}^{N} \int_{\mathcal{A}} T(\theta_t^i, U_i; (e(\overrightarrow{\theta}_t), \varphi_t), a) * \sigma(\theta_i^t, (e(\overrightarrow{\theta}_t), \varphi_t))[da] = \Pr\{\overrightarrow{\theta}_{t+1} \in U | \overrightarrow{\theta}_t, \varphi_t\} \tag{4.10}$$

The Cathedory extension theorem says that the definition of $P_N(\overrightarrow{\theta}_t, \varphi_t, U)$ for rectangular sets U can be uniquely extended to any $U \subset \mathcal{B}(\Theta^N)$. Note that the continuity of T and σ implies that P_N is continuous in S.

We use the mapping from nonanonymous states to anonymous state, $e : \Theta^N \to \Delta^N(\Theta)$, to translate the continuous Markov transition probability function over nonanonymous states, $P_N : \Theta^N \times \Psi \times \mathcal{B}(\Theta^N) \to [0,1]$, into a continuous Markov transition function over anonymous states, $P_N^A : \mathrm{S}_N \times \mathcal{B}(\mathrm{S}) \to [0,1]$. $\mathcal{B}(\mathrm{S})$ is the family of standard Borel sets over S. P_N^A can then be written for $V_1 \in \mathcal{B}(\Delta(\Theta)), V_2 \in \mathcal{B}(\Psi)$

$$\begin{aligned} P_N^A(\mathrm{s}{=}\,(\pi_t^\Theta, \varphi_t), (V_1, V_2)) &= P_N(\mathrm{e}^{-1}(\pi_t^\Theta), \varphi_t, e^{-1}(V_1 \cap \mathrm{S}_N)) * Y(V_2|\varphi) \\ &= \Pr\{\pi_{t+1}^\Theta \in V_1, \varphi_{t+1} \in V_2 | \pi_t^\Theta, \varphi_t\} = \\ &= \Pr\{\pi_{t+1}^\Theta \in V_1 | \pi_t^\Theta, \varphi_t\} * \Pr\{\varphi_{t+1} \in V_2 | \varphi_t\} \end{aligned} \tag{4.11}$$

From the Cathedory extension theorem, this description of P_N^A over products of Borel sets can be extended uniquely to a Markov transition function P_N^A over the full state space S_N. Note that the $t+1$ distributions of the idiosyncratic and aggregate certainty are independent conditional on s_t.

Our next theorem describes the ergodic properties of strategies of the large finite game. We will study the properties of the family of operators $\{P_N^A : \mathrm{S}_N \times \mathcal{B}(\mathrm{S}) \to [0,1]\}_{N=1}^\infty$ on the anonymous state of the economy induced by the type evolution operator T and a fixed, continuous Markov strategy played symmetrically by all agents in the N-agent game. We prove that for any finite τ, the behavior of the economy in periods $t+1$ through $t+\tau$ can be approximated by the continuum economy transition function for sufficiently large

N. Intuitively, as the number of agents in the economy becomes large, the idiosyncratic shocks experienced by each agent are smoothed out in the large economy. The proof of the theorem below uses our continuity and stochastic convergence results to make this intuition precise. We let $(P_N^A)^{\tau^*}(\mathrm{s}, \circ)$ denote the τ^* step-ahead measure over states given initial state s, and δ_{s} denote the measure that places an atom of weight 1 on state s.

Definition 12. *A function $f : X \to Y$ between metric space (X, d_X) and (Y, d_Y) is* ***finitely continuous*** *if there exists a finite partition Π of X such that for all $p \in \Pi$, f restrictet to p is uniformly continuous*

The notion of finite continuity extends the idea of continuity to a finite number of subsets of the domain of a function. Practically, this allows us to accomodate functions that admit discontinuities restricted to a non-generic set of the domain. For example, suppose a profit maximizing firm enters only if it's type, indicating production cost, is sufficiently low. This strategy could be easily written as a finitely discontinuous function partitioning the state space into "entry" and "exit" regions. Such a strategy could, at best, be written as a smoothed probability of entry in a continuous function of type.

Theorem 8. *Assume:*

- *T is continuous in $\mathrm{S} \times \mathcal{A}$ and finitely continuous in Θ*
- *$\sigma \in \Sigma$ is uniformly continuous in S and finitely continuous in Θ*
- *Fix $\tau^* < \infty$, $\delta > 0$, and $\rho \in (0, 1]$.*

If $s_t = \mathrm{s}_t^C$, then there exists N^* such that for economies with $N > N^*$ agents

$$d_{LP}((P_N^A)^{\tau^*}(\mathrm{s}(\omega, t), \circ), \delta_{(P^C)^{\tau^*}(\mathrm{s}(\omega,t))}) < \delta \tag{4.12}$$

for $\tau \in \{1, .., \tau^*\}$ with probability at least $1 - \rho$.

4.4. **Approximation Theorems.** We first use our results on the ergodic properties of the Stationary and Markov Perfect equilibria to show that for any $\varepsilon > 0, \rho \in [0, 1)$ the Stationary equilibria are $(\varepsilon, \rho)-$Markov Perfect Equilibria of a sufficiently large finite dynamic game. If we focus on the limit as the number of agents diverges to infinity, this result is similar to the claim that Stationary equilibria possess the Asymptotic Markov property of Benkard et al. [16].

Theorem 9. *Assume that:*

- *The type evolution operator T is state independent and has full support*
- *There is no aggregate uncertainty ($\Psi = \{\varphi\}$)*

Consider a Stationary Equilibrium of the continuum game $(\sigma^{SE}, \pi_{\infty}^{\Theta})$. *For any* $\varepsilon > 0$ *and* $\rho \in (0,1]$, *we can choose* $N^* < \infty$ *such that* σ^{SE} *is an* (ε, ρ)*-Markov Perfect Equilibrium of the large finite dynamic game for* $N > N^*$ *starting at* π_{∞}^{Θ}.

Theorem 9 cannot be generalized to generic cases when the type evolution operator is not state independent. The reason is that the stationary distribution could be an unstable equilibrium in the ergodic system defined by the large finite economy. In this case, stationarity in the continuum game does not imply ergodic stability for the large finite game.

The assumption of full support of the type evolution operator has important effects in the context of a Markov Perfect equilibrium of a dynamic economy. Since all states of economy are on the path of future play and the Markov strategies only respond to the present state of the economy, it is impossible for a model in our framework to capture certain forms of collusion. For example, firms in our model cannot engage in collusive behavior enforced by trigger strategies dictating everlasting punishments. The reason is that the state of the economy will, through random chance, eventually move out of the set of states indicating the punishment regime is in effect. However, we are able to capture the presence of temporary punishments. A leading example would be a price war model akin to Porter [71], although the end of the price war would be dictated by the type evolution operator as well as the agent strategies.

The following theorem shows that the exact MPE are approximate DCE of the continuum model. Theorem 8 proves that the behavior of the large finite economy can be approximated by the dynamical system of the DCE induced by the exact MPE strategy. As the MPE strategy is exactly optimal along the path of the MPE economy, it will be approximately optimal with high probability along the DCE economy's path. Although this theorem is of little interest directly, as it merely implies that difficult to compute MPE can be used to approximate more computationally tractable DCE, this theorem is an integral stepping stone to Theorem 11 that proves that the set of MPE is upper hemicontinuous in the limit as $N \to \infty$, with the limit set composed of the DCE of the continuum model.

Note that the N-agent σ_N^{MPE} is defined only over S_N. To state Theorem 10, we will extend the $N-$agent equilibrium strategy σ_N^{MPE} to a continuous function $\widetilde{\sigma}_N^{MPE} : \Theta \times \mathrm{S} \to \Delta(\mathcal{A})$ such that for all $\mathrm{s} \in \mathrm{S}_N$ we have $\widetilde{\sigma}_N^{MPE}(\theta, \mathrm{s}) = \sigma_N^{MPE}(\theta, \mathrm{s})$. To insure that this extension is well behaved, we will have to assume that σ_N^{MPE} is uniformly continuous in S and that the extension preserves this uniformity. From the definition of uniform continuity, for each $\varepsilon > 0$ we can choose $\delta(\varepsilon) > 0$ such that for all $\mathrm{s}, \widetilde{\mathrm{s}} \in \mathrm{S}_N$ such that $d_{\mathrm{S}}(\mathrm{s}, \widetilde{\mathrm{s}}) < \delta(\varepsilon)$ implies $|\sigma_N^{MPE}(\theta, \mathrm{s}) - \sigma_N^{MPE}(\theta, \widetilde{\mathrm{s}})| < \varepsilon$. We refer to $\delta(\varepsilon)$ as the *modulus of continuity* of

σ_N^{MPE}. We refer to an extension of σ_N^{MPE} to S as *modulus preserving* if there exists $C < \infty$ such the modulus of continuity of the extension, $\widetilde{\delta}(\varepsilon)$, obeys $\widetilde{\delta}(\varepsilon) < C * \delta(\varepsilon)$.

Theorem 10. *Consider a Markov Perfect Equilibrium strategy of the N-agent dynamic game, σ_N^{MPE}. Assume:*

- *T is continuous in $\mathrm{S} \times \mathcal{A}$ and finitely continuous in Θ*
- *$\sigma \in \Sigma$ is uniformly continuous in S and finitely continuous in Θ*
- *Fix $\tau^* < \infty$, $\delta > 0$, and $\rho \in (0, 1]$.*

For any $\varepsilon > 0$, we can choose $N^* < \infty$ such that if $N > N^*$:

(1) *Any uniformly continuous extension of σ_N^{MPE} from S_N to S that is modulus preserving is an $\varepsilon-$Dynamic Competitive Equilibrium for any initial state $\mathrm{s} \in \mathrm{S}$. Further, such an extension exists.*

(2) *Suppose S is compact[46] and $\Psi = \{\varphi\}$, then $(\widetilde{\sigma}_N^{MPE}, \mathrm{s}) \in \Sigma \times \mathrm{S}$ is an ε-Stationary Equilibrium of the nonatomic dynamic game for $N > N^*$ agents where for all $U \in \mathcal{B}(\Theta)$ where*

$$\pi_\infty^\Theta(U) = \int_{\mathcal{A}\times\Theta} T(\theta, U; \pi_\infty^\Theta, \widetilde{\sigma}_N^{MPE}(\theta, (\pi_\infty^\Theta, \varphi))[da]) * \pi_\infty^\Theta(d\theta)$$

and $\widetilde{\sigma}_N^{MPE}$ is a modulus preserving extension of σ_N^{MPE} from S_N to S.

The final theorem of our analysis of large dynamic games concerns the relationship between exact Markov Perfect equilibria and exact Stationary equilibria of the continuum game in terms of the realized actions for each type. If an econometrician estimates the stationary equilibria of a large finite economy, then theorem 11 implies that with high probability the exact Markov perfect equilibria would prescribe actions that are close in the action space to actions prescribed by the estimated stationary equilibrium.

Let the set of continuous Markov perfect equilibria of the N-agent stochastic game be denoted by the correspondence $\mathcal{E} : \mathbb{N} \rightrightarrows \Sigma$. Denote the equilibrium set of the nonatomic dynamic game as $\mathcal{E}^{NA}$. The following theorem assures us that MPE are not merely ε-DCE, but that as $\varepsilon \to 0$, the MPE will approach some DCE strategy.

Theorem 11. *Assume:*

[46] It would suffice that Θ is compact. Compactness is crucial for proving the existence of a stationary distribution, which relies on a fixed point argument. Alternative fixed point existence conditions can be employed if compactness is weakened. For example, it would also suffice if the map

$$F(\pi) = \int_{\mathcal{A}\times\Theta} T(\theta, U; \pi, \widetilde{\sigma}_N^{MPE}(\theta, (\pi, \varphi))[da]) * \pi(d\theta) \tag{4.13}$$

from $\Delta(\Theta)$ into $\Delta(\Theta)$ has a compact range.

- *T is continuous in* $\mathrm{S} \times \mathcal{A}$ *and finitely continuous in* Θ
- $\bigcup_{N=1}^{\infty} \mathcal{E}(N)$ *is uniformly equicontinuous in* S_N *and continuous in* Θ

Then the correspondence $\mathcal{E}$ is upper hemicontinuous with

$$\underset{N\to\infty}{Lim}\, \mathcal{E}(N) = \mathcal{E}^{\infty} \subset \mathcal{E}^{NA}$$

As in the case of Theorem 4, the intuition for Theorem 11 is that for the equilibrium correspondence $\mathcal{E}(N)$ to fail to be upper hemicontinuous, there would have to be a sequence of equilibria, $\{\sigma_N\}_{N=1}^{\infty}$, such that if the agents followed this strategy in the nonatomic dynamic game, a positive measure of the agents would have a significant profitable deviation. However, Theorem 10 proves that this is not possible. Our proof fomalizes this intuitive argument. We require the assumption that $\bigcup_{N=1}^{\infty} \mathcal{E}(N)$ is equicontinuous in order to show that a sequence of continuous equilibrium strategies of the large finite game converges to a continuous strategy in the nonatomic game.

The restriction that $\bigcup_{N=1}^{\infty} \mathcal{E}(N)$ be uniformly equicontinuous is difficult to discern apriori. The minimal assumption required to make us of our theorem is that there exists a convergent sequence of $\{\sigma_N : \sigma_N \in \mathcal{E}(N)\}$ and $\sigma_N \to \sigma_\infty$ such that σ_∞ is continuous. If the set $\bigcup_{N=1}^{\infty} \mathcal{E}(N)$ is *not* uniformly equicontinuous, then there may exist sequences of MPE that do not approach continuous DCE. It remains an open question as to whether any discontinuous DCE can approximate these equilibria for large N.[47]

As in the case of the static game, we can prove a simple corollary from the upper hemicontinuity result of Theorem 11. To so we need to assume the action space is a metric space of the form $(\mathcal{A}, d_{\mathcal{A}})$. Let the set of lotteries over $\mathcal{A}$ be a metric space under the Levy-Prokhorov metric, which defines a metric space $(\Delta(\mathcal{A}), d_{LP})$. Under these assumptions, we can state Theorem 11 in terms of metric convergence.

Theorem 12. *Assume:*

- *T is continuous in* $\mathrm{S} \times \mathcal{A}$ *and finitely continuous in* Θ
- $\bigcup_{N=1}^{\infty} \mathcal{E}(N)$ *is uniformly equicontinuous in* S_N *and continuous in* Θ
- *Consider* $\sigma_N^{MPE} \in \mathcal{E}(N)$

[47]The essential problem with proving the discontinuous DCE limit approximates the large finite MPE sequence is that the discontinuity in the limit may cause the ergodic properties of the large finite game to fail to resemble the dynamical orbit of the nonatomic limit. Nonpathological examples where this is the case can be easily constructed, but conditions under which this fails to occur is a nontrivial question.

For any $\delta > 0$, we can choose $N^* < \infty$ such that if $N > N^*$ there exists a DCE equilibrium strategy, σ^{DCE}, of the continuum game such that for all $\theta \in \Theta$ and $s \in S$ we have $|\sigma_N^{MPE}(\theta, \mathrm{s}) - \sigma^{DCE}(\theta, \mathrm{s})| < \delta$.

Theorem 11 shows that the actions taken by the agent in the MPE are approximated well by some DCE strategy of the nonatomic limit game. This result can substantially reduce the dimensionality of the parameter space required to solve or identify this model compositionally. First, note that an exact MPE is a function of the agent's own type and the current state of the economy. However, the state of the economy includes the distribution of agent types in the economy, which implies a curse of dimensionality occurs over a finite horizon since the state of the economy is an infinite dimensional variable in the limit as $N \to \infty$. However, in the DCE economy, given the state of the economy at time t, s_t, the state of the nonatomic limit economy at time $t + \tau$ is determined uniquely by the sequence of shocks $\{\varphi_{t+1}, ..., \varphi_{t+\tau}\}$. Therefore, optimal actions in a DCE can be written as functions of the initial state, s_t, the history of aggregate shocks between $t + 1$ and $t + \tau$, $\{\varphi_{t+1}, ..., \varphi_{t+\tau}\}$, and the agent's own type, $\theta_{t+\tau}$.

We will use the notation

$$\sigma^{DCE}(\circ|\mathrm{s}_t) : \Theta \times \bigcup_{\tau=0}^{\infty} \Psi^\tau \to \Delta(\mathcal{A})$$

where $\bigcup_{\tau=0}^{\infty} \Psi^\tau$ represents the potential sets of future aggregate shocks. As noted in Benkard et al. [17], under this notation σ^{DCE} is not formally Markovian in that it depends on payoff irrelevant information, the history of aggregate shocks and a past state of the economy. This formulation has the benefit that the state space of the strategy, $\bigcup_{\tau=0}^{\infty} \Psi^\tau$, is not a function of the number of agents in the economy and no curse of dimensionality problem results as $N \to \infty$. For the purposes of identification and computational tractability, some assumption must be made regarding the influence of past shocks. Benkard et al. [17] propose either an exponentially decayed weighting scheme or that only a finite history of aggregate shocks influences the present period's action. Either assumption limits the influence of the past on the present period's action and reduces the asymptotically infinite dimension MPE strategy space to a tractable, finite dimension space of functions. Of course, any such scheme is an approximation of the DCE formally defined as an approximation in our theorems above, and our work has provided no assurance that any such approximation scheme is asymptotically consistent.[48]

[48]Benkard et al. [17], in a model of dynamic industry competition, and Krussel and Smith [47], in a macroeconomic context, provide evidence that in some modeling contexts reducing an infinite dimensional

One form of DCE of particular interest is the SE defined above. A SE requires both a lack of aggregate shocks and a stationary state of the economy. As SE strategies are functions from the type space to the action space, the equilibrium strategies can be defined by a finite dimensional function regardless of the number of agents in the economy or the time horizon for prediction. Therefore, if the stationarity and no aggregate uncertainty assumptions are satisfied by the data, an econometrician can use a low dimensional model to estimate models of the behavior of industries with an arbitrarily large number of participants. Further, computing policy experiments by estimating counterfactual SE states is also computationally tractable.

4.5. **A Note on Macroeconomics.** Numerous modern macroeconomic models use large dynamic games as microfoundations for the phenomena of interest. The analysis often proceeds by assuming a continuum of nonatomic agents that take the market aggregates as immutable when solving their individual optimization problems. The use of a continuum of nonatomic agents is a plausible technique for approximating real economies that involve a large, but finite, number of agents interacting to effect equilibrium outcomes. In the vast majority of these studies, the use of this approximation is neither questioned nor tested.

Although it is beyond the scope of this paper to investigate particular macroeconomic models, the analysis conducted above in the context of the Ericson and Pakes framework could be readily extended to assess whether macroeconomic models satisfy the requisite continuity properties. If the continuity properties are satisfied, then the model is likely an adequate approximation of the underlying behavior of the agents in the economy. In the event that the continuity property is not satisfied, care ought to be taken when interpreting the results of the analysis.

5. Conclusion

Large finite mechanisms have become important in several different branches of the economics literature, but the central intuitions underlying the relation to their nonatomic analogs has not been expressed in a fashion suitable for applied theoreticians and econometricians prior to this work. We have shown that in a general mechanism design framework, weak continuity and semi-anonymity restrictions are sufficient for the nonatomic mechanisms to approximate the outcomes of their large finite analogs. Further, we are able to show that as the number of agents increases, the approximation improves in both the strategy and payoff space of the model.

problem to a finite dimensional problem will result in little loss of predictive accuracy. The extent to which these suggestive results would extend to the contexts we treat is an interesting topic for future research.

One reason that the study of the limit behavior of large finite mechanisms has been of increasing interest to microtheorists is the need to design well-functioning markets. A leading example of such a design task was the FCC spectrum auction. Nonatomic models are often easier to analyze than their large finite equivalents, principally due to the ease with which equilibria can be found when the agents treat market aggregates as exogenous to their own decisions. If a candidate market's nonatomic form meets the desiderata of the designer and the market outcomes are continuous in the distribution of agent actions, then the analyst can have confidence that the large finite implementation of the market will almost achieve these properties.

The limit properties of large finite markets are also of interest as models of the strategic microfoundations for nonatomic economies such as Walrasian equilibria in exchange economies. The search for satisfactory underpinnings for general equilibrium models is almost as old as the notion of general equilibrium itself with early suggestions such as Walras's tâtonnement process. This search continues today with studies that employ the sophisticated tools of modern game theory such as Reny and Perry [74].[49] The theorems in this work provide a simple set of tools for providing analyses of the relationship between large finite and nonatomic markets, which will be of use to economists studying markets wherein price-taking provides crucial tractability but where the model is sufficiently complex that providing a formal analysis of the behavior using game-theoretic techniques is intractable. While we leave studies of this sort for future work, we note that this form of analysis could potentially be of particular interest to macroeconomists concerned with whether a price taking equilibrium can be founded on the decisions of strategically interacting individual agents.

Our study of the uniform price auction with arbitrary preferences for multiple units of heterogeneous goods is novel and has resisted prior efforts due to the difficulty of analyzing the complex strategies of agents with complementary valuations for successive units. However, when price taking behavior is assumed of the bidders a general equilibrium model results, and the complementary valuations do not pose a tractability issue for showing either the existence or efficiency of the final outcome. Since price taking behavior is approximately optimal in the limit, the results we prove for the nonatomic uniform price auction, such as the truthfulness[50] and efficiency of the allocation and price, hold approximately for

[49]Reny and Perry [74] take a fundamentally different approach to their analysis by finding exact equilibria for each game as $N \to \infty$. We conjecture that our framework would provide many of the same results within a less arduous analysis structure.

[50]The equilibrium is unique up to outcome equivalent deviations in the agent strategies from truthful behavior.

large auctions. These results are reassuring, since it is known that small uniform price auctions may be neither truthful nor efficient due to the incentive for agents to withhold demand to manipulate the closing price of the auction. Large uniform price auctions are frequently used to allocate items such as U.S. Treasury Bills and spectrum licenses, which makes understanding the properties of the outcome under general preference structures crucially important.

The second novel application concerned the behavior of the Markov perfect Equilibria of stochastic games of interest to econometricians, industrial organization economists, and labor economists. The large parameter space of these games has made estimating or computationally solving models with more than a handful of agents intractable. However, we are able to show that the Markov perfect equilibria of large stochastic games can in many cases be approximated by the dynamic competitive equilibria of nonatomic Walrasian models. In the case of stationary models, the Markov perfect equilibria can be well approximated by stationary equilibria. Stationary equilibria have the practical advantage of being of fixed, finite dimension regardless of the number of agents. Potential applications include the study of policy to regulate large markets or to analyze the anti-trust implications of mergers in industries with a large number of participants.

Finally, our paper suggests that earlier works studying the convergence of large finite markets to competitive equilibria relied on symmetry and continuity. In an appendix, we demonstrate this by using our framework to generalize the McLean and Postlewaite's [51] model of mechanisms for aggregating diffuse information as well as the leader-follower model of Fudenberg, Levine, and Pesendorfer [31]. Given that we have identified the crucial continuity condition required for these markets to be well behaved as the market grows large, we are able to generate many of the results of these papers under more general assumptions using simpler proof techniques. In addition, it is clear that much of the prior analysis of these problems was focused on developing the necessary continuity properties without identifying them as such. In addition to demonstrating the power of the tools and techniques we have developed, the analysis we provide emphasizes a deeper intuition as to what makes these markets function well in the limit as $N \to \infty$.

Interesting topics for future work include further exploring the effects of markets that become ex post in the limit as $N \to \infty$. This implies a number of desirable properties for the mechanism in terms of simplicity for agents to learn and play. In addition, a number of large markets have been implemented that have resisted satisfying equilibrium analysis, a leading example of which are one-to-one and many-to-one matching markets. The framework outlined above could provide a powerful tool for analyzing these problems.

References

[1] Al-Najjar, N. (2004) "Aggregation and the Law of Large Numbers in Large Economies," *Games and Economic Behavior*, Vol. 47, pp. 1-35.

[2] Al-Najjar, N. (2008) "Large Games and the Law of Large Numbers," *Games and Economic Behavior*, Vol. 64, pp. 1-34.

[3] Al-Najjar, N. and R. Smordinsky (2000) "Pivotal Players and the Characterization of Influence," *Journal of Economic Theory*, Vol. 92, pp. 318-342.

[4] Alos-Ferrer, Carlos (1999) "Dynamical Systems with a Continuum of Randomly Matched Agents," *Journal of Economic Theory*, Vol. 86, pp. 245-267.

[5] Athey, S. (1995) "Characterizing Properties of Stochastic Objective Functions," MIT Working Paper No. 96-1.

[6] Athey, S. (2001) "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, pp. 861-889.

[7] Aumann, R.J. (1964) "Markets with a Continuum of Traders," *Econometrica*, 32, pp. 39-50.

[8] Aumann, R.J. (1966) "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica*, 34, pp. 1-17.

[9] Aumann, R.J. and A. Brandenburger (1995) "Epistemic Conditions for Nash Equilibrium," *Econometrica*, Vol. 63, pp. 1161-1180.

[10] Aumann, R.J. (1999) "Interactive Epistemology I: Knowledge," *International Journal of Game Theory*, 28, pp. 263-300.

[11] Ausubel, L. and P. Crampton (2002) "Demand Reduction and Inefficiency in Multi-Unit Auctions," *mimeo.*

[12] Ausubel, L. and R. Deneckere (1993) "A Generalized Theorem of the Maximum," *Economic Theory*, 3, pp. 99 - 107.

[13] Back, K. and J. Zender (1993) "Auctions of Divisible Goods: On the Rationale for the Treasury Experiment," *The Review of Financial Studies*, 6, pp. 733 - 764.

[14] Backus, M. and G. Lewis (2009) "An Estimable Demand System for a Large Auction Platform Market," *mimeo.*

[15] Benkard, C.L.; B. Van Roy; and G.Y. Weintraub (2009) "Industry Dynamics: Foundations for Models with an Infinite Number of Firms," *mimeo.*

[16] Benkard, C.L.; B. Van Roy; and G.Y. Weintraub (2009) "Markov Perfect Industry Dynamics With Many Firms," *Econometrica*, 76, pp. 1375 - 1411.

[17] Benkard, C.L.; B. Van Roy; and G.Y. Weintraub (2009) "Computational Methods for Oblivious Equilibrium," *mimeo.*

[18] Berge, C. (1963) *Topological Spaces*, The Macmillan Company: New York.

[19] Bergemann, D. and S. Morris (2005) "Robust Mechanism Design," *Econometrica*, 73, pp. 1771-1813.

[20] Bergin, J. and D. Bernhardt (1995) "Anonymous Sequential Games: Existence and Characterization of Equilibrium," *Economic Theory*, pp 461 - 489.

[21] Billingsley, P. (1968) *Convergence of Probability Measures*, John Wiley & Sons, Inc.: New York.

[22] Bonnans, J. and A. Shapiro (2000) *Perturbation Analysis of Optimization Problems*, Springer-Verlag Inc.: New York.

[23] Aliprantis, C. and K. Border (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide, Third edition*, Springer-Verlag Inc.: New York.

[24] Budish, E. (2008) "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes," *mimeo.*

[25] Carmona (2004) "Nash Equilibria of Games with a Continuum of Players," *mimeo.*

[26] Carmona, G. and K. Podczeck (2009) "On the Existence of Pure-Strategy Equilibria in Large Games," *Journal of Economic Theory*, p.1300 - 1319.

[27] Cripps, M. and J. Swinkels (2006) "Efficiency of Large Double Auctions," *Econometrica*, 74, pp. 47-92.

[28] Ergin, H. and T. Sonmez (2006) "Games of School Choice Under the Boston Mechanism," *Journal of Public Economics*, 90, pp. 215-237.

[29] Ericson, R. and A. Pakes (1995) "Markov-Perfect Industry Dynamics: A Framework for Empirical Work," *Review of Economic Studies*, 62, pp. 53-82.

[30] Fan, K. (1952) "Fixed-Point and Minimax Theorems in Locally Convex Topological Linear Spaces," *Proceedings of the National Academy of Sciences*, 38, pp. 121-126.

[31] Fudenberg, D.; D. Levine; and W. Pesendorfer (1998) "When are Nonanonymous Players Negligible?" *Journal of Economic Theory*, 79, pp. 46-71.

[32] Futia, C. (1982) "Invariant Distributions and the Limiting Behavior of Markovian Economic Models," *Econometrica*, Vol. 50, pp. 377-408.

[33] Gale, D. (2000) *Strategic Foundations of General Equilibrium*, Cambridge University Press: Cambridge, U.K.

[34] Gibbs, A.I. and F.E. Su (2002) "On Choosing and Bounding Probability Metrics," *International Statistical Review*, 70, pp. 418-435.

[35] Green, E.J. (1980) "Noncooperative Price Taking in Large Dynamic Markets," *Journal of Economic Theory*, 22, pp. 155-181.

[36] Gretsky, N.; J. Ostroy; and W. Zane (1992) "The Nonatomic Assignment Model," *Economic Theory,* 2, pp. 103 - 127.

[37] Grossman, S. and J. Stiglitz (1976) "Information and Competitive Price Systems," *American Economic Review*, 66, pp. 246-253.

[38] Gul, F. and A. Postlewaite (1992) "Asymptotic Efficiency in Large Exchange Economies," *Econometrica*, 60, pp. 1273 - 1292.

[39] Hildenbrand, W. (1974) *Core and Equilibria of a Large Economy*, Princeton University Press: Princeton, New Jersey.

[40] Hammond, P. and Y. Sun (2006) "Joint Measurability and the One-Way Fubini Property For a Continuum of Random Variables," *Proceedings of the American Mathematical Society*, 134, pp. 737-747.

[41] Immorlica, N. and M. Mahdian (2005) "Marriage, Honesty, and Stability," *SODA 2005*, pp. 53-62.

[42] Jackson, M. and A. Manelli (1997) "Approximately Competitive Equilibria in Large Finite Economies," *Journal of Economic Theory*, 77, pp. 354 - 376.

[43] Kalai, E. (2005) "Large Robust Games," *Econometrica*, 72, pp. 1631-1665.

[44] Khan, M. and Sun, Y. (1997) "Non-Cooperative Games with Many Players," *Handbook of Game Theory with Economic Applications, Vol. 3*, North Holland: New York.

[45] Kojima, F. and P. Pathak (2007) "Incentives and Stability in Large Two Sided Matching Markets," *mimeo.*

[46] Kranton, R. and D. Minehart (2001) "A Theory of Buyer-Seller Networks," *American Economic Review*, 91, pp. 485-508.

[47] Krussel, P. and A. Smith (1998) "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106, pp. 867-896.

[48] Leininger, W. (1984) "A Generalization of the 'Maximum Theorem'," *Economics Letters*, 15 , pp. 309-313.

[49] Levine, D.P. and W. Pesendorfer (1995) "When are Agents Negligible," *American Economic Review*, 85, pp. 1160-1170.

[50] Luenberger, D. (1984) *Linear and Nonlinear Programming*, Springer: New York.

[51] McLean, R. and A. Postlewaite (2002) "Informational Size and Incentive Compatibility," *Econometrica*, 70, pp. 2421-2453.

[52] Manea, M. (2008) "Asymptotic Ordinal Inefficiency of Random Serial Dictatorship," *mimeo.*

[53] Mas-Colell, A. (1984) "On a Theorem of Schmeidler," *Journal of Mathematical Economics*, 13, pp. 201-206.

[54] Mas-Colell, A. and X. Vives (1993) "Implementation in Economies with a Continuum of Agents," *The Review of Economic Studies*, 60, pp. 613-629.

[55] Massart, P. (1990) "The Tight Constant in the Dvoretzky-Kiefer-Wolfowitz Inequality," *The Annals of Probability*, 18, pp. 1269 - 1283.

[56] McAdams (2006) "Monotone Equilibrium in Multi-Unit Auctions," *The Review of Economc Studies*, 73, pp. 1039 - 1056.

[57] Meyn, S.P. and R.L. Tweedie (1993) *Markov Chains and Stochastic Stability*, Springer-Verlag: London and New York.

[58] Milgrom, P.R. (2004) *Putting Auction Theory to Work*. Cambridge University Press, New York.

[59] Milgrom, P. and C. Shannon (1994) "Monotone Comparative Statics," *Econometrica*, Vol. 62, pp. 157-180.

[60] Milgrom, P. and R. Weber (1985) "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, 10, pp. 619-632.

[61] Muller, A. and D. Stoyan (2002) *Comparison Methods for Stochastic Models and Risks.* New York: John Wiley and Sons, Inc.

[62] Nisan, N. and I. Segal (2005) "The Communication Requirements of Efficient Allocations and Supporting Prices," *Journal of Economic Theory*, 129, pp. 192-224.

[63] Nobel, A. and A. Dembo (1993) "A Note on Uniform Laws of Averages for Dependent Processes," *Statistics and Probability Letters*, 17, pp. 169-172.

[64] Osborne, M. and A. Rubinstein (1994) *A Course in Game Theory*, Massachusetts Institute of Technology Press: Cambridge, Massachusetts.

[65] Otani, Y. and J. Sicilian (1990) "Limit Properties of Equilibrium Allocations of Walrasian Strategic Games," *Journal of Economic Theory*,51, pp. 295-312.

[66] Robert, D. and P. Milgrom (2008) "Core-selecting Package Auctions," *International Journal of Game Theory*, 36, pp. 393-407.

[67] Palfrey, T.R. and S. Srivastava (1986) "Private Information in Large Economies," *Journal of Economic Theory*, 39, pp. 34-58.

[68] Parkes, D.; J. Kalagnanam; and M. Eso (2002) "Achieving Budget-Balance with Vickrey Based Payment Schemes in Exchanges," *Proceedings of the 17th International Joint Conference on Artificial Intelligence (IJCAI-01):* pp. 1161 - 1168.

[69] Pesendorfer, W. and J. Swinkels (2000) "Efficiency and Information Aggregation in Auctions," *American Economic Review*, 90, pp. 499-525.

[70] Petrov, V. (1995) *Limit Theorems of Probability Theory: Sequence of Independent Random Variables*, Oxford University Press: Oxford, England.

[71] Porter, R. (1983) "A Study of Cartel Stability: The Joint Executive Committee, 1880 - 1886," *The Bell Journal of Economics*, 14, pp. 301-314.

[72] Postlewaite, A. and D. Schmeidler (1978) "Approximate Efficiency of Non-Walrasian Nash Equilibria," *Econometrica*, pp. 127 - 135.

[73] Rachev, S.T.; V. Haynatzka; and G. Haynatzki (1997) "Probability Metric and Limit Theorems in AIDS Epidemiology" in *Real and Stochastic Analysis: Recent Advances* (ed. M.M. Rao), pp.159-215.

[74] Reny, P. and M. Perry (2006) "Toward a Strategic Foundation for Rational Expectations Equilibrium," *Econometrica*, 74, pp. 1231 - 1269.

[75] Roberts, D. and A. Postlewaite (1976) "The Incentives for Price-Taking Behavior in Large Exchange Economies," *Econometrica*, 44, pp. 115 - 127.

[76] Roth, A. (1982) "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7, pp. 617-628.

[77] Roth, A. and M. Sotomayor (1990) *Two Sided Matching: a Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press: Cambridge, England.

[78] Roth, A.; U. Rothblum; and J. Vande Vate (1993) "Stable Matchings, Optimal Assignments, and Linear Programming," *Mathematics of Operations Research*, 18, pp. 803-828.

[79] Rubinstein, A. (1986) "Finite Automata Play the Repeated Prisoner's Dilemma," *Journal of Economic Theory*, 39, pp. 83-96.

[80] Rustichini, A.; M. Satterthwaite; and S. Williams (1994) "Convergence to Efficiency in a Simple Market with Incomplete Information," *Econometrica*, 62, pp. 1041-1063.

[81] Sabourian, H. (1990) "Anonymous Repeated Games with a Large Number of Players and Random Outcomes," *Econometrica*, 61, pp. 325-351.

[82] Satterthwaite, M. and A. Shneyerov (2007) "Dynamic Matching, Two-Sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition," *Econometrica*, 75, pp. 155-200.

[83] Satterthwaite, M. and S. Williams (2002) "The Optimality of a Simple Market Mechanism," *Econometrica*, 70, pp. 1841 - 1863.

[84] Schmeidler, D. (1973) "Equilibrium Points of Nonatomic Games," *Journal of Statistical Physics*, 7, pp. 295-300.

[85] Segal, I. (1999) "Contracting with Externalities," *The Quarterly Journal of Economics*, 114, pp. 337 - 388.

[86] Serrano, R. (2002) "Decentralized Information and the Walrasian Outcome: A Pairwise Meetings Market with Private Values," *Journal of Mathematical Economics*, 38, pp. 65-89.

[87] Swinkels, J. (2001) "Efficiency of Large Private Value Auctions," *Econometrica*, 69, pp. 37-68.

[88] Topsøe, Flemming; Richard Dudley; and Jørgen Hoffman-Jørgensen (1976) "Two Examples Concerning Uniform Convergence of Measures w.r.t. Balls in Banach Spaces," *Empirical Distributions and Processes, Lecture Notes in Mathematics,* 566, pp. 141-146.

[89] van der Vaart, A. and J. Wellner (1996) *Weak Convergence and Empirical Processes*, Springer-Verlag: New York.

[90] Vidossich, G. (1969) "A Theorem on Uniformly Continuous Extensions of Mappings Defined in Finite Dimensional Spaces," *Israel Journal of Mathematics*, 7, pp. 207 - 210.

[91] Wolinsky, A. (1988) "Dynamic Markets with Competitive Bidding," *The Review of Economic Studies,* 55, pp. 71-84.

6. Appendix A: Proofs

6.1. **Uniform Glivenko-Cantelli Classes.** We begin this section with some useful results from the theory of the weak convergence of empirical processes.[51] We are able to use this theory to extend the classical Glivenko-Cantelli theorem to finite dimensional Euclidean spaces, which implies convergence of empirical distributions to the underlying true distribution as $N \to \infty$. We are able to show that this implies weak-* convergence of the empirical distribution to the true distribution and provide an asymptotic convergence rate. First we will define the notion of an empirical distribution.

Definition 13. *Given a set* $\{X_1, ..., X_N\}$ *of independent realizations of a random variable* $X : \Omega \to T$ *with distribution* P*, define the associated* ***empirical measure***

$$\mathbb{P}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i} \tag{6.1}$$

where δ_{X_i} *is the measure that places weight* 1 *on the element* X_i*.*

In this study we will focus on cases wherein the space T is a subset of the finite dimensional Euclidean space $\mathbb{R}^d$, although other applications might require more general sets T. In order to describe uniformity over a a family of measurable (with respect to Q) functions $\mathcal{F} = \{f : T \to \mathbb{R}\}$, we define the norm

$$\|Q\|_{\mathcal{F}} = \sup\{f \in \mathcal{F} : |Qf| = \int |f(t) * Q(dt)|\} \tag{6.2}$$

Our asymptotics will focus on families of functions, $\mathcal{F}$, for which the expectation under the empirical measure converges to the expectation under the true measure uniformly over the

[51] Readers desiring more information on these topics shoudl reference van der Vaart and Wellner [89]

family of functions. Families of such functions are referred to as uniform Glivenko-Cantelli classes.

Definition 14. *A family of measurable functions $\mathcal{F} = \{f : T \to \mathbb{R}\}$ is a **Uniform Glivenko-Cantelli** class if for all $f \in \mathcal{F}$ we have $\|\mathbb{P}_N - P\|_{\mathcal{F}} \to 0$ as $N \to \infty$.*

We will be particularly concerned with families of functions defined by indicator functions over sets. In this setting, a class of sets $\mathcal{C} = \{C \subset T\}$ is identified with $\mathcal{F}_C = \{f_C : f_C(t) = 1\{t \in C\}$ for some $C \in \mathcal{C}\}$.[52]

Theorem 13. *(van der Vaart et al. [89], p. 135) A class of sets $\mathcal{C}$ is a uniform Glivenko-Cantelli class if and only if it forms a Vapnik-Červonenkis (VC) Class*

There are numerous methods for identifying VC Classes, most notably through *shattering* conditions that define the *VC index* of the class $\mathcal{C}$. In lieu of providing references for formal techniques for identifying VC Classes,[53] we will focus on a subset of the family of lower contours of $\mathbb{R}^d$, sets of the form $L^R(q) = \{p \in \mathbb{R}^d : p \leq q\}$. The Cumulative Distribution Function (CDF) of a measure π_0 can then be defined as $F(q) = \pi_0(L^R(q))$. We will make use of the following metric over the space of measures over $\mathbb{R}^d$.

Definition 15. *Consider two cumulative distribution functions F, G over the state space $\Omega \subset \mathbb{R}^d$. The **Kolmogorov (Uniform) metric** is $d_K(F,G) = \sup_{x \in \Omega} |F(x) - G(x)|$*

Given that the set L is a VC Class, we can obtain the following theorem.

Theorem 14. *Consider a random variable $X : \Omega \to \mathbb{R}^d$, $d < \infty$, with measure π_0 and associated CDF $F(y) = \int 1\{x \leq y\} * \pi_0(dx)$. For N i.i.d. realizations, $\{X_1, ..., X_N\}$ drawn from π_0, denote the empirical CDF $F_N(y) = \frac{1}{N}\sum_{i=1} 1\{X_i \leq y\}$. Then we have almost surely*

$$d_K(\pi_N, \pi_0) = \sup_{y \in \mathbb{R}^d} |F_N(y) - F(y)| \to 0 \tag{6.3}$$

Proof. (Proof of Theorem 14) Since the sets of the form $\{x : x \leq y\}$ for $y \in \mathbb{R}^d$ are lower contours and hence form a VC Class, we have that the class of functions $\mathcal{F} = \{f : f_y(x) = 1\{x \leq y\}\}$ is a Uniform Glivenko-Cantelli class, and so uniformly across $y \in \mathbb{R}^d$ we have as $N \to \infty$ almost surely

$$F_N(y) = \frac{1}{N}\sum_{i=1} 1\{X_i \leq y\} \to \int 1\{x \leq y\} * \pi_0(dx) = F(y) \tag{6.4}$$

[52] Let $1\{E\}$ refer to the indicator of event E.

[53] The interested reader is referred to van der Vaart et al [89] chapter 2 for references. Proofs that the lower contours are VC classes (amongst other examples) is provided in that text.

This can then be re-written

$$d_K(\pi_N, \pi_0) = \sup_{y \in \mathbb{R}^d} |F_N(y) - F(y)| \to 0 \tag{6.5}$$

□

Corollary 4. *Define the empirical measure generated by the counting measure over $\{X_1, ..., X_N\}$ as π_N. Then $\pi_N \to \pi_0$ almost surely in the weak-* topology over $\Delta(\mathbb{R}^d)$*

Proof. (Proof of Corollary 4) From Billingsley (p. 18, [21]) we have that $F_N(y) \to F(y)$ at continuity points of F implies $\pi_N \to \pi_0$ in the weak-* topology. Since we have uniform convergence $F_N(y) \to F(y)$ for all y almost surely, we have $\pi_N \to \pi_0$ in the weak-* topology in the weak-* topology. □

Corollary 5. *Consider a random variable $X : \Omega \to \mathbb{R}^d$, $d < \infty$ associated CDF $F(y)$. Denote the N realization empirical CDF as $F_N(y)$. Then*

$$\Pr\{\sqrt{N} \sup_{y \in \mathbb{R}^d} |F_N(y) - F(y)| > t\} = C * e^{-2t^2}$$

where the constant $C > 0$ depends only on the dimension d of the support of X.

Proof. (Proof of Corollary 5) This result follows directly from Theorems 2.6.7 and 2.14.9 of van der Vaart and Wellner [89]. □

6.2. **Proofs of Theorems in Main Body.**

Proof. (of Theorem 1) Given $u(\theta, \pi_0^\Theta, g(\circ, \circ))$ is continuous, Mas-Colell [53] shows that there exists a measure τ over the space $\Theta \times \mathcal{M}$ such that

$$\tau(\{(\theta, m) : \forall m' \in \mathcal{M}, u(\theta, \pi_0^\Theta, g(\pi_0^\mathcal{M}, m)) \geq u(\theta, \pi_0^\Theta, g(\pi_0^\mathcal{M}, m'))\}) = 1 \tag{6.6}$$

Since we have assumed that Θ and $\mathcal{M}$ are subspaces of finite dimensional Euclidean spaces, the measure μ generates such a conditional probability measure $\tau_\mathcal{M}(\cdot|\theta)$[54] over the message space $\mathcal{M}$ that defines an equilibrium in distributional strategies for almost all $\theta \in \Theta$. This strategy is obviously symmetric across agents. We will denote this equilibrium distributional strategy $m^\infty : \Theta \to \Delta(\mathcal{M})$.

In order to convert the distributional strategies into pure strategies, consider $\theta \in \Theta$ and the associated distribution strategy $m^\infty(\theta)$.Let $F^\theta : \mathcal{M} \to [0, 1]$ denote the cumulative distribution function (CDF) of the distribution $m^\infty(\theta)$. If F^θ is nonatomic, then for an

[54]This implicitly assumes that the equilibrium measure μ over $\Theta \times \mathcal{M}$ has an associated conditional probability measure. Since we have assumed that Θ and $\mathcal{M}$ are subspaces of finite dimensional Euclidean spaces, the measure μ generates such a conditional probability measure (see Durrett for [?] details).

agent of type $(\theta, p) \in \Theta^E$, let the pure strategy in the extended space be $m^E(\theta, p) = (F^\theta)^{-1}(p)$. If F^θ has atoms and $p \notin (F^\theta)^{-1}([0,1])$, then let $m^E(\theta, p) = (F^\theta)^{-1}(\overline{p})$ where

$$\overline{p} = \min\{p' : p' \geq p \text{ and } p \in (F^\theta)^{-1}([0,1])\} \tag{6.7}$$

Otherwise, let $m^E(\theta, p) = (F^\theta)^{-1}(p)$. Clearly $m^E(\theta, p)$ generates the same marginal distribution over the message space as $m(\theta)$. Since $m(\theta)$ met the Nash incentive constraints, so does $m^E(\theta, p)$ over the extended type space. □

Proof. (of Theorem 2) Denote the empirical distribution of agent types and messages, respectively, as π_N^Θ and $\pi_N^\mathcal{M}$. From our Theorem 14, we have $\pi_N^\Theta \to \pi_0^\Theta$ and $\pi_N^\mathcal{M} \to \pi_0^\mathcal{M}$ almost surely in the Kolmogorov Metric as $N \to \infty$. Consider $\omega \in \Omega$ such that $\pi_N^\mathcal{M}(\omega) \to \pi_0^\mathcal{M}$ and $\pi_N^\Theta(\omega) \to \pi_0^\Theta$ (we will drop the dependence on ω for notational cleanliness).

From the uniform pointwise convergence of u_N to u, for any $\varepsilon > 0$ we can choose N sufficiently large that

$$\sup_{(\theta, \pi^\Theta, x) \in \Theta \times \Delta(\Theta) \times \mathcal{X}} |u_N(\theta, \pi^\Theta, x) - u(\theta, \pi^\Theta, x)| < \varepsilon \tag{6.8}$$

For the duration of the proof we will move between u_N and u without including the ε factor in this step of our approximation, which can be made arbitrarily small for sufficiently large N.

Consider $\varepsilon > 0$. From the uniform convergence of the sequence $\{g_n(\circ)\}_{n=1}^\infty$, we can choose N_1 such that for all $\theta \in \Theta$, $\pi^\Theta \in \Delta(\Theta)$, $\pi^\mathcal{M} \in \Delta(\mathcal{M})$, and $m \in \mathcal{M}$

$$|u(\theta, \pi^\Theta, g(\pi^\mathcal{M}, m)) - u(\theta, \pi^\Theta, g_{N_1}(\pi^\mathcal{M}, m))| < \frac{\varepsilon}{2} \tag{6.9}$$

From the uniform equicontinuity of $\{g(m, \cdot)\}_{m \in \mathcal{M}}$ in a relatively open set around $\pi_0^\mathcal{M}$ and the uniform equicontinuity of $\{u(\theta, \cdot, \cdot)\}_{\theta \in \Theta}$, we can find for each $\delta^* > 0$ such that if $d_K(\pi_N^\mathcal{M}, \pi_0^\mathcal{M}) < \delta^*$ and $d_K(\pi_N^\Theta, \pi_0^\Theta) < \delta^*$, then we have for any $\theta \in \Theta$, $m \in \mathcal{M}$

$$|u(\theta, \pi_N^\Theta, g(\pi_N^\mathcal{M}, m)) - u(\theta, \pi_0^\Theta, g(\pi_0^\mathcal{M}, m))| < \frac{\varepsilon}{2} \tag{6.10}$$

Since $\pi_N^\mathcal{M} \to \pi_0^\mathcal{M}$ and $\pi_N^\Theta \to \pi_0^\Theta$, there exists an N_2 such that for $N > N_2$ we have $d_K(\pi_N^\mathcal{M}, \pi_0^\mathcal{M}) < \frac{\delta^*}{2}$ and $d_K(\pi_N^\Theta, \pi_0^\Theta) < \delta^*$. Obviously we can choose N_3 such that for all $N > N_3$ we have for all $m, \widetilde{m} \in \mathcal{M}$

$$d_K(\pi_N^\mathcal{M}, \pi_N^\mathcal{M} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m]) < \frac{\delta^*}{2} \tag{6.11}$$

In this case $\mathrm{d}_K(\pi_0^\mathcal{M}, \pi_N^\mathcal{M} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m]) < \delta^*$.

Therefore, for $N > \max\{N_1, N_2, N_3\}$ we have for all $\theta \in \Theta$ and $m, \widetilde{m} \in \mathcal{M}$

$$
\begin{aligned}
(6.12)\quad & |u(\theta,\pi_N^\Theta, g_N(\pi_N^{\mathcal{M}} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], m)) - u(\theta, \pi_0^\Theta, g(\pi_0^{\mathcal{M}}, m))| \leq \\
& |u(\theta, \pi_N^\Theta, g_N(\pi_N^{\mathcal{M}} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], m)) - u(\theta, \pi_N^\Theta, g(\pi_N^{\mathcal{M}} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], m))| + \\
& |u(\theta, \pi_N^\Theta, g(\pi_N^{\mathcal{M}} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], m)) - u(\theta, \pi_0^\Theta, g(\pi_0^{\mathcal{M}}, m))| \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

But then this implies that for all $\widetilde{m} \in \mathcal{M}$

$$
(6.13)\qquad |u(\theta, \pi_N^\Theta, g_N(\pi_N^{\mathcal{M}} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], \widetilde{m})) - u(\theta, \pi_0^\Theta, g(\pi_0^{\mathcal{M}}, \widetilde{m}))| < \varepsilon
$$

Therefore any $m \in$supp$[m^\infty(\theta)]$ where

$$
(6.14)\qquad m^\infty(\theta) \in \arg\max_{\widetilde{m} \in \mathcal{M}} u(\theta, \pi_0^\Theta, g(\pi_0^{\mathcal{M}}, \widetilde{m}))
$$

is an ε optima of

$$
(6.15)\qquad \max_{\widetilde{m} \in \mathcal{M}} u(\theta, \pi_N^\Theta, g_N(\pi_N^{\mathcal{M}} + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], \widetilde{m}))
$$

Given that the above argument holds for all $\omega \in \Omega$ such that $\pi_N^{\mathcal{M}}(\omega) \to \pi_0^{\mathcal{M}}$ and that the later convergence is almost sure with respect to P, we have that an ex-post $\varepsilon-$Pure Strategy Nash equilibrium is realized almost surely asymptotically. Therefore, for any $\rho > 0$ there is a finite N_ρ such that the subset of Ω such that $d_K(\pi_N^\Theta, \pi_0^\Theta) < \delta^*$ and $d_K(\pi_N^{\mathcal{M}}, \pi_0^{\mathcal{M}}) < \delta^*$ has measure at least $1 - \rho$. By letting $N^* = \max\{N, N_\rho\}$ we have our result. □

Proof. (of Theorem 3) By the same logic employed in the proof of Theorem 2, we know that the K draw empirical distribution of messages induced by the strategy $m^N(\circ)$, denoted π_K^N (with an associated empirical CDF over the message space denoted $F^K(\cdot)$), converges almost surely to the true distribution π_∞^N (with CDF $F^\infty(\cdot)$) in the Kolmogorov metric on the space of measures as $K \to \infty$.[55] Consider $\omega \in \Omega$ such that $\pi_K^N(\omega) \to \pi_\infty^N$ as $K \to \infty$ (we will drop the dependence on ω for notational cleanliness). Note that the nonatomic form of the mechanism is then $g(\pi_\infty^N, \cdot)$.

[55]Note that we are not asserting that m^n remains an equilibrium when we take the number of agent to ∞.

From the uniform pointwise convergence of u_N to u, for any $\varepsilon > 0$ we can choose N sufficiently large that

$$\sup_{(\theta,\pi^{\Theta},x)\in\Theta\times\Delta(\Theta)\times\mathcal{X}} |u_N(\theta,\pi^{\Theta},x) - u(\theta,\pi^{\Theta},x)| < \varepsilon \tag{6.16}$$

For the duration of the proof we will move between u_N and u without including the ε factor in this step of our approximation, which can be made arbitrarily small for sufficiently large N.

From the definition of Bayesian Nash equilibrium we have for all $\widetilde{m} \in \mathcal{M}$ and for all $m \in$supp$[m^N(\theta)]$

$$E^P[u(\theta,\pi_N^{\Theta}, g_N(\pi_N^N, m)) - u(\theta,\pi_N^{\Theta}, g_N(\pi_N^N + \frac{1}{N}[\delta_{\widetilde{m}} - \delta_m], \widetilde{m}))] \geq 0 \tag{6.17}$$

From Theorem 5 we know that

$$P(\sqrt{N}\sup_{m\in\mathcal{M}} |F^N(m^N) - F^{\infty}(m)| > \lambda) \leq Ce^{-2\lambda^2} \tag{6.18}$$

with the choice of C independent of the form of the equilibrium strategy. Let

$$M = \sup_{x\in\mathcal{X},\theta\in\Theta,\pi^{\Theta}\in\Delta(\Theta)} u(\theta,\pi^{\Theta},x) - \inf_{x\in\mathcal{X},\theta\in\Theta,\pi^{\Theta}\in\Delta(\Theta)} u(\theta,\pi^{\Theta},x) \tag{6.19}$$

which is finite from the boundedness of the utility function.

Since $g_N(\circ, m)$ uniformly (over m) converges pointwise to $g(\circ, m)$ as $N \to \infty$, we can choose $N_1 > 0$ such that for all $N > N_1$ we have for all $\theta \in \Theta$, $m \in \mathcal{M}$

$$|u(\theta,\pi_{\infty}^{\Theta}, g_N(\pi_{\infty}^N, m)) - u(\theta,\pi_{\infty}^{\Theta}, g(\pi_{\infty}^N, m))| < \frac{\varepsilon}{2} \tag{6.20}$$

Since $\{g(m,\cdot)\}_{m\in\mathcal{M}}$ is uniformly equicontinuous in a relatively open set around $\pi_0^{\mathcal{M}}$, and $\{u(\theta,\cdot,\cdot)\}_{\theta\in\Theta}$ is uniformly equicontinuous, then for any $\varepsilon > 0$ we can choose $\lambda > 0$ so that if $d_K(\pi_K^{\Theta}, \pi_{\infty}^{\Theta})$, $d_K(\pi_K^N, \pi_{\infty}^N) < \lambda$ we have uniformly over $m \in \mathcal{M}$

$$|u(\theta,\pi_{\infty}^{\Theta}, g(\pi_{\infty}^N, m)) - u(\theta,\pi_K^{\Theta}, g(\pi_K^N, m))]| < \frac{\varepsilon}{2} \tag{6.21}$$

Therefore

$$|u(\theta,\pi_{\infty}^{\Theta}, g(\pi_{\infty}^N, m)) - u(\theta,\pi_K^{\Theta}, g_N(\pi_K^N, m))]| < \varepsilon \tag{6.22}$$

Using our convergence rate result, for any $\lambda > 0$ we can choose N_2 such that for $K > \max\{N_1, N_2\}$ we have for all $m \in$supp$[m^N(\theta)]$

$$|u(\theta,\pi_{\infty}^{\Theta}, g(\pi_{\infty}^N, m)) - E^P[u(\theta,\pi_K^{\Theta}, g_N(\pi_K^N, m))]| < (1 - Ce^{-2\lambda^2}) * \varepsilon + Ce^{-2\lambda^2} * M \tag{6.23}$$

$$|u(\theta,\pi_\infty^\Theta,g(\pi_\infty^N,\widetilde{m}))-E^P[u(\theta,\pi_K^\Theta,g_N(\pi_K^N+\frac{1}{K}[\delta_{\widetilde{m}}-\delta_m],\widetilde{m}))]|< (1-Ce^{-2\lambda^2})*\varepsilon+\varepsilon+Ce^{-2\lambda^2}*M \tag{6.24}$$

where the additional ε in the second relation is a result of of $\pi_K^N+\frac{1}{K}[\delta_{\widetilde{m}}-\delta_m]\to\pi_K^N$ in the Kolmogorov metric as $K\to\infty$ and the continuity assumptions imposed on $\{g(m,\cdot)\}_{m\in\mathcal{M}}$ and $\{u(\theta,\cdot,\cdot)\}_{\theta\in\Theta}$. For $\lambda>0$ sufficiently small these relations can be written

$$|u(\theta,\pi_\infty^\Theta,g(\pi_\infty^N,m))-E^P[u(\theta,\pi_K^\Theta,g_N(\pi_K^N,m))]|<2*\varepsilon \tag{6.25}$$

$$|u(\theta,\pi_\infty^\Theta,g(\pi_\infty^N,\widetilde{m}))-E^P[u(\theta,\pi_K^\Theta,g_N(\pi_K^N+\frac{1}{K}[\delta_{\widetilde{m}}-\delta_m],\widetilde{m}))]|<3*\varepsilon \tag{6.26}$$

Substituting these relations into the Bayesian Nash equilibrium condition yields for all $m\in$supp$[m^N(\theta)]$ and $\widetilde{m}\in\mathcal{M}$

$$u(\theta,\pi_\infty^\Theta,g(\pi_\infty^N,m))-u(\theta,\pi_\infty^\Theta,g_N(\pi_\infty^N,\widetilde{m}))\geq-5*\varepsilon \tag{6.27}$$

For any choice of $\varepsilon>0$ we can choose N sufficiently large that for all $m\in$supp$[m^N(\theta)]$ and $\widetilde{m}\in\mathcal{M}$

$$u(\theta,\pi_\infty^\Theta,g_N(\pi_\infty^N,m))+5*\varepsilon\geq u(\theta,\pi_\infty^\Theta,g_N(\pi_\infty^N,m')) \tag{6.28}$$

Therefore $m^N(\theta)$ is an $\varepsilon-$Nash Equilibrium of the nonatomic game generated by the strategy $m^N(\theta)$. □

Proof. (of Theorem 4) Consider a sequence of exact Bayesian-Nash equilibrium strategies, $\{m^N:\Theta\to\Delta(\mathcal{M})\}_{N=1}^\infty$ where $m^N\in\mathcal{E}(N)$. For all $M\in\mathcal{B}(\mathcal{M})$, let

$$\pi_\infty^N(M) = \int_\Theta \Pr\{m^N(\theta)\in M\}\pi_0^\Theta(d\theta) \tag{6.29}$$

$$\pi_\infty^\infty(M) = \int_\Theta \Pr\{m^\infty(\theta)\in M\}\pi_0^\Theta(d\theta) \tag{6.30}$$

Suppose $m^N\to m^\infty\notin\mathcal{E}^{NA}$. This can be the case only if there exists $\rho,\varepsilon>0$ such that for all $N^*>0$ there exists $N>N^*$ such that for a measure ρ of types of agents in the nonatomic game such that for all $m\in$supp$[m^\infty(\theta)]$ the following holds

$$u(\theta,\pi_0^\Theta,g_N(\pi_\infty^\infty,m))+\varepsilon<\sup_{m'\in\mathcal{M}}u(\theta,\pi_0^\Theta,g_N(\pi_\infty^\infty,m')) \tag{6.31}$$

From the continuity of u and g and the convergence of m^N to m^∞ for any $\varepsilon>0$ we can choose $\delta>0$ and N large enough that if $d_{LP}(\pi_\infty^\infty,\pi_\infty^N)<\delta$, then for all $m\in$supp$[m^N(\theta)]$

and m'∈supp$[m^{\infty}(\theta)]$

$$|u(\theta,\pi_{\infty}^{\Theta},g_N(\pi_{\infty}^{\infty},m))-u(\theta,\pi_{\infty}^{\Theta},g_N(\pi_{\infty}^{\infty},m'))|<\varepsilon \tag{6.32}$$

Note that as

$$\sup_{\theta\in\Theta}|m^N(\theta)-m^{\infty}(\theta)|\to 0 \tag{6.33}$$

we have that $\pi_{\infty}^{N}\to\pi_{\infty}^{\infty}$ in the weak-* topology. Therefore, for any $\delta>0$ we can choose N large enough that $d_{LP}(\pi_{\infty}^{\infty},\pi_{\infty}^{N})<\delta$. Finally Theorem 3 implies for all $m\in$supp$[m^N(\theta)]$

$$u(\theta,\pi_{\infty}^{\Theta},g_N(\pi_{\infty}^{N},m))+\varepsilon>\sup_{m'\in\mathcal{M}}u(\theta,\pi_{\infty}^{\Theta},g_N(\pi_{\infty}^{N},m')) \tag{6.34}$$

But this implies for sufficiently large N that there cannot be an ε better response in the nonatomic game when the agents play m^{∞}. Since this holds for all $\varepsilon>0$, it must be the case that $m^{\infty}\in\mathcal{E}^{NA}$. Since this holds for all such sequences of strategies, we have from Theorem 17.16 of Aliprantis and Border [23] that the equilibrium correspondence is upper hemicontinuous. □

Proof. (Proof of Theorem 6) Since the space $\Delta(\mathcal{M})$ is first countable, it suffices to show continuity with respect to sequences in $\Delta(\mathcal{M})$. Consider an arbitrary sequence $\{\pi_t^{\mathcal{M}}\}_{t=1}^{\infty}$ such that $\pi_t^{\mathcal{M}}$ converges to $\pi_0^{\mathcal{M}}$ in the Kolmogorov metric and an arbitrary $\varepsilon>0$. In addition, consider and arbitrary continuous function $f:\mathcal{X}\to\mathbb{R}$.

From the boundedness of $\mathcal{X}$ we can define $M=\sup_{x,x'\in\mathcal{X}}d_{\mathcal{X}}(x,x')<\infty$. From the assumption that the discontinuities of $\mathcal{G}$ are diffuse, we can claim that there exists a neighborhood $\mathcal{U}_T$ of $\pi_0^{\mathcal{M}}$ such that $\pi_T^{\mathcal{M}}\in\mathcal{U}_T$ and a measure of at least $1-\frac{\varepsilon}{2M}$ of the elements of $\mathcal{G}$,denoted $\mathcal{G}_C$ with corresponding index set $\Lambda_C\subset\Lambda$, are continuous in this neighborhood. The elements of $\mathcal{G}_C$ are equicontinuous, implying that we can choose $T^*\geq T$ such that for all $g_\alpha\in\mathcal{G}_C$, $d_{\mathcal{X}}(f(g_\alpha(\pi_{T^*}^{\mathcal{M}},m)),f(g_\alpha(\pi_0^{\mathcal{M}},m)))<\frac{\varepsilon}{2}$.[56] Combining these relations we have

$$|E[f(g(\pi_{T^*}^{\mathcal{M}},m))-f(g(\pi_0^{\mathcal{M}},m))]|=|\int[f(g_\alpha(\pi_{T^*}^{\mathcal{M}},m))-f(g_\alpha(\pi_0^{\mathcal{M}},m))]Q(d\alpha)| \tag{6.35}$$

$$\begin{aligned}&\leq\ |\int_{\Lambda_C}[f(g_\alpha(\pi_{T^*}^{\mathcal{M}},m))-f(g_\alpha(\pi_0^{\mathcal{M}},m))]Q(d\alpha)|+M*\frac{\varepsilon}{2M}\\&\leq\ \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

[56]In most of the other proofs of this work, the symbol N is reserved to refer to the number of agents in the economy. In this proof, the symbol N denotes an index to the sequence of measures $\{\pi_n^{\mathcal{M}}\}_{n=1}^{\infty}$. The usage of N within this proof will not be used in the remainder of the work.

This implies that for any fixed m the convergence of $\pi_t^{\mathcal{M}}$ to $\pi_0^{\mathcal{M}}$ entails convergence of $g(\pi_t^{\mathcal{M}}, m)$ to $g(\pi_0^{\mathcal{M}}, m)$ in the weak-* topology. $\square$

Proof. (of Lemma 1)Consider a measure of declared demand curves $\pi^{\mathcal{M}}, \nu^{\mathcal{M}} \in \Delta(\mathcal{M})$ such that $d_k(\pi^{\mathcal{M}}, \nu^{\mathcal{M}}) < \delta$. Since we assumed that demand curves are decreasing, we can define the aggregate demand in terms of K upper contour sets of the form $\mathcal{M}(x,p) = \{q \in \mathcal{M} : q(p) \geq x\}$. Since we know that upper contour sets from a Vapnik-Chervonenkis class, we can conclude that $d_k(\pi^{\mathcal{M}}, \nu^{\mathcal{M}}) < \delta$ implies for all $x \in \{0, ..., K\}$ and for all $p \in [0, \overline{v}]$, we have $|\pi^{\mathcal{M}}(\mathcal{M}(x,p)) - \nu^{\mathcal{M}}(\mathcal{M}(x,p)) < \delta$. This implies that aggregate demand,

$$D(\pi^{\mathcal{M}}, p) = \int q(p) * \pi^{\mathcal{M}}(dq) \tag{6.36}$$

is continuous in the Kolmogorov metric over $\Delta(\mathcal{M})$. Since $D(\pi^{\mathcal{M}}, p)$ is decreasing in p, it is almost everywhere continuous in p.

Fix $\pi^{\mathcal{M}} \in \Delta(\mathcal{M})$ and r. Suppose there exists some $\varepsilon > 0$ such that for all $\delta > 0$ and $\nu^{\mathcal{M}} \in \Delta(\mathcal{M})$ such that $d_k(\pi^{\mathcal{M}}, \nu^{\mathcal{M}}) < \delta$ we have

$$|p(\pi^{\mathcal{M}}; r) - p(\nu^{\mathcal{M}}; r)| > \varepsilon \tag{6.37}$$

Without loss of generality, we assume $p(\pi^{\mathcal{M}}; r) > p(\nu^{\mathcal{M}}; r)$. As $\delta \to 0$, for all p we have $D(\pi^{\mathcal{M}}, p) \to D(\nu^{\mathcal{M}}, p)$, which implies from our Market Clearing conditions that $D(\pi^{\mathcal{M}}, p)$ must be discontinuous at $p(\pi^{\mathcal{M}}; r)$. But then this occurs if and only if there is an atom of agents declaring a price schedule with a discontinuity at $p(\pi^{\mathcal{M}}; r)$. For generic r it will not be the case that

$$D(\pi^{\mathcal{M}}, p) = r > \underset{\gamma \to 0}{Lim} D(\pi^{\mathcal{M}}, p - \delta) \tag{6.38}$$

and hence we neglect this case.[57] But then consider the remaining case, wherein for all r' in a neighborhood of r we have

$$D(\pi^{\mathcal{M}}, p) > r' > \underset{\gamma \to 0}{Lim} D(\pi^{\mathcal{M}}, p - \delta) \tag{6.39}$$

But in this case, for $\delta > 0$ sufficiently small, if $d_k(\pi^{\mathcal{M}}, \nu^{\mathcal{M}}) < \delta$ then we have both

$$D(\nu^{\mathcal{M}}, p) \;>\; r > \underset{\gamma \to 0}{Lim} D(\nu^{\mathcal{M}}, p - \delta) \tag{6.40}$$

$$D(\nu^{\mathcal{M}}, p) \;>\; r' > \underset{\gamma \to 0}{Lim} D(\nu^{\mathcal{M}}, p - \delta) \tag{6.41}$$

[57] This will occur for a measure 0 set of supplies, r, as this random variable is distributed nonatomically.

But then this implies $p(\pi^{\mathcal{M}}; r) = p(\nu^{\mathcal{M}}; r)$. From this contradiction, we see $p(\pi^{\mathcal{M}}; r)$ is continuous except at the measure 0 set of r wherein

$$D(\pi^{\mathcal{M}}, p) = r > \underset{\gamma \to 0}{Lim} D(\pi^{\mathcal{M}}, p - \delta) \tag{6.42}$$

Therefore, we conclude that $p(\pi^{\mathcal{M}})$ is continuous in the Kolmogorov metric on both the domain and range. We strengthen our result to uniform continuity by noting that the compactness of $\mathcal{M}$ implies that $\Delta(\mathcal{M})$ is compact, and the Heine-Cantor theorem implies that $p(\pi^{\mathcal{M}})$ is uniformly continuous.

It remains to show continuity of the allocation functions, $x(\pi^{\mathcal{M}}, q)$, with respect to the Kolmogorov topology on $\Delta(\mathcal{M})$. Since $q(p)$ is weakly decreasing in price, we can write

$$\{r : x(\pi^{\mathcal{M}}, q; r) = x\} = \{p(\pi^{\mathcal{M}}; r) : \ q(p(\pi^{\mathcal{M}}; r)) \geq x\} \cap \{p(\pi^{\mathcal{M}}; r) : q(p(\pi^{\mathcal{M}}; r)) \leq x\} \tag{6.43}$$

$\{p(\pi^{\mathcal{M}}; r) : q(p(\pi^{\mathcal{M}}; r)) \geq x\}$ defines an upper contour in the space of prices, while $\{p(\pi^{\mathcal{M}}; r) : q(p(\pi^{\mathcal{M}}; r)) \leq x\}$ defines a lower contour set in this space. Since $p(\pi^{\mathcal{M}})$ is continuous in the Kolmogorov topology on the domain and range, the probability of both $\{p(\pi^{\mathcal{M}}; r) : q(p(\pi^{\mathcal{M}}; r)) \geq x\}$ and $\{p(\pi^{\mathcal{M}}; r) : q(p(\pi^{\mathcal{M}}; r)) \leq x\}$ (and hence their intersection) are uniformly continuous in $\pi^{\mathcal{M}}$ for any such choice of price interval.[58] Therefore, for any $\varepsilon > 0$ we can choose $\delta > 0$ such that if $d_k(\pi^{\mathcal{M}}, \nu^{\mathcal{M}}) < \delta$ then

$$|\Pr\{x(\pi^{\mathcal{M}}, q) = x\} - \Pr\{x(\nu^{\mathcal{M}}, q) = x\}| < \varepsilon \tag{6.44}$$

But then this simply implies equicontinuity of $x(\cdot, q)$ in the Kolmogorov metric. We can again strengthen equicontinuity to uniform equicontinuity using the Heine-Cantor theorem.

It remains to show that realized agent utility

$$E[u(\theta, \omega, x(\pi^{\mathcal{M}}, m))|\theta, p] - p(\pi^{\mathcal{M}}, m) \cdot x \tag{6.45}$$

is upper semicontinuous in $\Delta(\mathcal{M}) \times \mathcal{M}$ where $\Delta(\mathcal{M})$ is endowed with the weak-* topology and $\mathcal{M}$ with the Euclidean norm. Note for two measure $\pi^{\mathcal{M}}, \nu^{\mathcal{M}} \in \Delta(\mathcal{M})$ we have $\pi^{\mathcal{M}} \to \nu^{\mathcal{M}}$ in the weak-* topology but not the Kolmogorov topology if there is a discontinuity in the CDF of $\pi^{\mathcal{M}}$ that moves as $\pi^{\mathcal{M}} \to \nu^{\mathcal{M}}$, which corresponds to an atom in $\pi^{\mathcal{M}}$ with changing support in the limit. However, a small shift in the support of an atom causes a small shift in the aggregate demand, and hence we have $p(\pi^{\mathcal{M}}; r) \to p(\nu^{\mathcal{M}}; r)$. This implies that the only discontinuity can lie in $x(\pi^{\mathcal{M}}, m) \to x(\nu^{\mathcal{M}}, m)$. In fact, from the definition of $x(\pi^{\mathcal{M}}, m)$ in terms of upper contours of price, it is clear that continuity of x generically requires the Kolmogorov topology. Note from the market clearing condition

[58] This is another point wherein we use the uniformity of the Kolmogorov metric to our advantage.

however, that x can only jump upwards as $\pi^{\mathcal{M}} \to \nu^{\mathcal{M}}$. This implies that $x(\pi^{\mathcal{M}}, m)$ is upper semicontinuous in $\pi^{\mathcal{M}}$ in the weak-* topology. Since we require that agents only declare demands below their true interim demand, agent utility can only increase due to this sudden addition of a unit to his allocation. This implies agent utility is upper semicontinuous in $\pi^{\mathcal{M}}$. Upper semicontinuity with respect to $\mathcal{M}$ follows by an identical argument that discontinuities in $x(\pi^{\mathcal{M}}, m)$ can only imcrease agent utility. □

Proof. (of Lemma 2) First we will prove that the agent demand schedule declarations must be increasing in θ, and then from this monotonicity show that $p(\omega, r)$ must be increasing in ω and decreasing in r in equilibrium. This in turn implies that the equilibrium price distribution is fully revealing and that the equilibrium is efficient.

Lemma 4. *$D_\theta(p)$ must be weakly increasing in θ in equilibrium. Further, $\theta \geq \widetilde{\theta}$ and $\theta \neq \widetilde{\theta}$ implies there exists some p such that $D_\theta(p) > D_{\widetilde{\theta}'}(p)$.*

Proof. Consider the problem facing an agent considering what quantity to demand given their own type θ and a market clearing price $p = p(\omega, r)$. Note that market price, agent type, and the state of the economy are jointly distributed in equilibrium according to some PDF $g(\theta, p, \omega)$. Since the market clearing price is exogenous to a single agent's decision, we have that θ and p are independent conditional on ω.

We will show that interim agent utility given (θ, p) exhibits increasing differences for *any* equilibrium price function $p(\omega, r)$. Consider

$$E[u(\theta, \omega, x)|\theta, p] \tag{6.46}$$

which is an expectation with respect to ω given $(\theta, p = p(\omega, r))$. We can write

$$g(\omega|\theta, p) = \frac{g(\omega, p, \theta)}{g(\theta, p)} = \frac{g(p|\omega) * f(\theta|\omega) * g(\omega)}{g(\theta, p)} \tag{6.47}$$

Consider $\theta \geq \theta'$ and $\omega \geq \omega'$. Then we can write

$$\frac{g(\omega|\theta, p)}{g(\omega|\theta', p)} = \frac{f(\theta|\omega)}{f(\theta'|\omega)} * \frac{g(\theta', p)}{g(\theta, p)} \tag{6.48}$$

Assumption 8 that $f(\theta|\omega)$ is log supermodular implies that

$$\begin{aligned} \frac{g(\omega|\theta, p)}{g(\omega|\theta', p)} &= \frac{f(\theta|\omega)}{f(\theta'|\omega)} * \frac{g(\theta', p)}{g(\theta, p)} \\ &> \frac{f(\theta|\omega')}{f(\theta'|\omega')} * \frac{g(\theta', p)}{g(\theta, p)} = \frac{g(\omega'|\theta, p)}{g(\omega'|\theta', p)} \end{aligned} \tag{6.49}$$

Therefore $g(\omega|\theta, p)$ is log supermodular in (ω, θ, p). If $k > l$ and $\theta \geq \theta'$ we have

$$E[u(\theta, \omega, k) - u(\theta, \omega, l)||\theta, p] > E[u(\theta', \omega, k) - u(\theta', \omega, l)||\theta', p] \tag{6.50}$$

since $u(\theta, \omega, k) - u(\theta, \omega, l)$ is strictly increasing in both θ and ω from assumption 6. Since $E[u(\theta, \omega, k)||\theta, p]$ has increasing differences in (θ, x), we have from Milgrom and Shannon [59] that

$$D_\theta(p) = \underset{x \in \{0,1,\ldots,K\}}{\arg\max} \; E[u(x, \omega, \theta)|\theta, p] - p \cdot x \tag{6.51}$$

is weakly increasing in θ. From the fact that $E[u(x, \omega, \theta)|\theta, p]$ is strictly increasing in θ, a straightforward revealed preference argument shows that $\theta \geq \widetilde{\theta}$ and $\theta \neq \widetilde{\theta}$ implies there exists some p such that $D_\theta(p) > D_{\widetilde{\theta}'}(p)$. □

First we will show that aggregate demand

$$D(p) = \int D_\theta(p) * \pi_0^\Theta(\omega)[d\theta] \tag{6.52}$$

is strictly increasing in ω at all prices p. Let $\Theta(k, p) = \{\theta : D_\theta(p) \geq k\}$, wihch we note is an upper contour in Θ as $D_\theta(p)$ is increasing in θ. Suppose that $\omega > \omega'$, which implies that $\pi_0^\Theta(\omega)$ is greater than $\pi_0^\Theta(\omega')$ in the strong stochastic order (see Milgrom [66] for a proof). Then we have

$$\pi_0^\Theta(\omega)[\Theta(k, p)] > \pi_0^\Theta(\omega')[\Theta(k, p)] \tag{6.53}$$

But we have that

$$D(p; \omega) = \textstyle\sum_{k=1}^{K} \pi_0^\Theta(\omega)[\Theta(k, p)] \tag{6.54}$$

Thus we have $\omega > \omega'$ implies that $D(p; \omega) > D(p; \omega')$. Since the distribution of demand curves is distributed nonatomically (as $\pi_0^\Theta(\omega)$ is nonatomic), $D(p; \omega)$ is continuous in p in equilibrium. From our market clearing condition and the fact that $\pi_0^\Theta(\omega)[d\theta]$ is nonatomic we have in equilibrium

$$p(\omega, r) = \sup_{p \in [0, \overline{v}]} p \text{ such that } \int D_\theta(p) * \pi_0^\Theta(\omega)[d\theta] = r$$

and hence $p(\omega, r)$ is strictly increasing in ω and decreasing in r in equilibrium. Therefore, given an ex post realization of $p(\omega, r)$ and r, the state of the economy is fully revealed. □

Proof. (of Lemma 3) First note that the decision problem facing the agent in the Private Values case at the interim stage is

$$\text{(Private Values)} \qquad D_\theta(p) = \underset{x \in \{0,1,\dots,K\}}{\arg\max}\ u(x,\omega,\theta) - p \cdot x$$

No expectation is required since in the private value setting, there is no additional information contained in the market clearing price. The first welfare theorem then reveals that the outcome is Pareto efficient, which in the quasi-linear case impliex ex post efficiency.

The agents declare their demand truthfully after conditioning on the information conveyed by the equilibrium price. Formally, the agents declare their expected ex post utility truthfully

$$\text{(Common Value)} \qquad D_\theta(p) = \underset{x \in \{0,1,\dots,K\}}{\arg\max}\ E[u(x,\omega,\theta)|\mathcal{P}^*] - p \cdot x$$

From the first welfare theorem, since each agent receives $D_\theta(p(\omega, r))$, the outcome is efficient with respect to these informationally constrained utility assesments. Therefore, the outcome is Pareto efficient with respect to the interim demands calculated in (Common Value). In the quasi-linear utility setting we use, this is equivalent to informational efficiency with respect to $\mathcal{P}^*$. □

Proof. (of Theorem 7) Consider the **private values case** and fix $\varepsilon > 0$. Theorem 4 shows that for any $\varepsilon > 0$ we can choose N^* such that for $N > N^*$ we have

$$\sup_{\theta \in \Theta} |m^N(\theta) - m^\infty(\theta)| < \varepsilon \tag{6.55}$$

where $m^\infty(\theta) = \theta$ is the truthful declaration. This allocation will be exactly efficient if we map an agent of type θ to $m^N(\theta)$. Therefore, we have

$$\int u_N(m^N(\theta), \omega, x_N(\pi^{\mathcal{M}}, m^N(\theta))) * \pi^\Theta(d\theta) = \max_{x^* \in \mathcal{J}} \frac{1}{N} \sum_{i=1}^{N} u_N(m^N(\theta_i), \omega, x^*(i)) \tag{6.56}$$

Buit from the continuity of u with respect to θ, the uniform convergence of u_N to u, and the fact that $|m^N(\theta) - \theta| \to 0$ uniformly as $N \to \infty$ we have for large enough N

$$\text{For all } \theta,\ |u_N(m^N(\theta), \omega, x_N(\pi^{\mathcal{M}}, m^N(\theta))) - u_N(\theta, \omega, x_N(\pi^{\mathcal{M}}, m^N(\theta)))| < \varepsilon \tag{6.57}$$

This in turn implies

$$\int u_N(\theta, \omega, x_N(\pi^{\mathcal{M}}, m^N(\theta))) * \pi^\Theta(d\theta) + \varepsilon \geq \max_{x^* \in \mathcal{J}} \frac{1}{N} \sum_{i=1}^{N} u_N(\theta_i, \omega, x^*(i)) \tag{6.58}$$

as required.

In order to prove that our result holds in the **common value case**, we will use the fact the nonatomic equilibrium strategies, $D_\theta(p)$, are strictly increasing in θ, the strategy set is upper hemicontinuous, and the type distribution is nonatomic. For any $M \in \mathcal{B}(\mathcal{M})$ let

$$\pi_N^{\mathcal{M}}(M) = \int_\Theta \Pr\{m^N(\theta) \in M\}\pi_0^\Theta(d\theta) \tag{6.59}$$

$$\pi_0^{\mathcal{M}}(M) = \int_\Theta \Pr\{m^\infty(\theta) \in M\}\pi_0^\Theta(d\theta) \tag{6.60}$$

where $m^\infty(\theta) = D_\theta(\circ)$ is the equilibrium declaration in the nonatomic game.

Lemma 5. *$\pi_N^{\mathcal{M}} \to \pi_0^{\mathcal{M}}$ in the Kolmogorov metric.*

Proof. First note that $D_\theta(p)$ can be represented as a set of price discontinuities, which implies that we can take $\mathcal{M} = [0, \overline{v}]^K = \Theta$. Let $M(\theta^*) = \{\theta : \theta \leq \theta^*\}$ be a lower countour set. Then Kolmogorov convergence requires that for any $\varepsilon > 0$ there exists N^* so that for all $N > N^*$ we have uniformly over sets $M(\theta^*) = \{\theta : \theta \leq \theta^*\}$

$$|\pi_N^{\mathcal{M}}(M(\theta^*)) - \pi_0^{\mathcal{M}}(M(\theta^*))| < \varepsilon \tag{6.61}$$

Since $\pi_0^\Theta(\omega)$ is absolutely continuous with respect to the Lebesgue measure and $m^\infty(\theta)$ is strictly increasing, for any $\varepsilon > 0$ we can choose $\delta > 0$ so that

$$\pi_0^{\mathcal{M}}(M(\theta^*)) - \pi_0^{\mathcal{M}}(M(\theta^* - \delta)) < \varepsilon \tag{6.62}$$

$$\pi_0^{\mathcal{M}}(M(\theta^* + \delta)) - \pi_0^{\mathcal{M}}(M(\theta^*)) < \varepsilon \tag{6.63}$$

Then choose N^* sufficiently large that

$$\sup_{\theta \in \Xi} |m^N(\theta) - m^\infty(\theta)| < \delta \tag{6.64}$$

Then we have

$$\pi_N^{\mathcal{M}}(M(\theta^*)) \leq \pi_0^{\mathcal{M}}(M(\theta^* + \delta)) \leq \pi_0^{\mathcal{M}}(M(\theta^*)) + \varepsilon \tag{6.65}$$

$$\pi_N^{\mathcal{M}}(M(\theta^*)) \geq \pi_0^{\mathcal{M}}(M(\theta^* - \delta)) \geq \pi_N^{\mathcal{M}}(M(\theta^*)) - \varepsilon \tag{6.66}$$

Together these imply $|\pi_N^{\mathcal{M}}(M(\theta^*)) - \pi_0^{\mathcal{M}}(M(\theta^*))| < \varepsilon$ as desired. Therefore $\pi_N^{\mathcal{M}} \to \pi_0^{\mathcal{M}}$ in the Kolmogorov metric. □

Consider the ex post utility in the large finite game

$$u_N(\theta, \pi^\Theta, x_N(\pi^{\mathcal{M}}, m^N(\theta))) \tag{6.67}$$

From the uniform convergence of u_N to u and x_N to x, we have for large enough N

$$|u(\theta, \pi^{\Theta}, x(\pi^{\mathcal{M}}, m^N(\theta))) - u_N(\theta, \pi^{\Theta}, x_N(\pi^{\mathcal{M}}, m^N(\theta)))| < \frac{\varepsilon}{4} \tag{6.68}$$

Given continuity of u with respect to $\Delta(\Theta)$ and x with respect to $\Delta(\mathcal{M})$ and the almost sure convergence of $\pi^{\Theta} \to \pi_0^{\Theta}$ and $\pi^{\mathcal{M}} \to \pi_N^{\mathcal{M}}$, for any $\rho > 0$ we can choose N sufficiently large that

$$|u(\theta, \pi^{\Theta}, x(\pi^{\mathcal{M}}, m^N(\theta))) - u(\theta, \pi_0^{\Theta}, x(\pi_N^{\mathcal{M}}, m^N(\theta)))| < \frac{\varepsilon}{4} \tag{6.69}$$

with probability at least $1 - \rho$. From our lemma above, $\pi_N^{\mathcal{M}} \to \pi_0^{\mathcal{M}}$ in the Kolmogorov metric which implies for N sufficiently large we have

$$|u(\theta, \pi_0^{\Theta}, x(\pi_0^{\mathcal{M}}, m^N(\theta))) - u(\theta, \pi_0^{\Theta}, x(\pi_N^{\mathcal{M}}, m^N(\theta)))| < \frac{\varepsilon}{4} \tag{6.70}$$

with probability at least $1 - \rho$. From the convergence of $m^N(\theta) \to m^{\infty}(\theta)$ and the continuity of u, we have

$$|u(m^N(\theta), \pi_0^{\Theta}, x(\pi_N^{\mathcal{M}}, m^N(\theta))) - u(\theta, \pi_0^{\Theta}, x(\pi_0^{\mathcal{M}}, m^N(\theta)))| < \frac{\varepsilon}{4} \tag{6.71}$$

with probability at least $1 - \rho$. Note that

$$u(m^N(\theta), \pi_0^{\Theta}, x(\pi_0^{\mathcal{M}}, m^N(\theta))) = u(\theta, \pi_0^{\Theta}, x(\pi_0^{\mathcal{M}}, \theta)) \tag{6.72}$$

Then we have putting all of these relations together that for large enough N

$$|u(\theta, \pi_0^{\Theta}, x(\pi_0^{\mathcal{M}}, \theta)) - u_N(\theta, \pi^{\Theta}, x_N(\pi^{\mathcal{M}}, m^N(\theta)))| < \varepsilon \tag{6.73}$$

with probability at least $1 - p$. But note that this them implies for any filtration $\mathcal{P}^*$ of $\Omega \times (0, K) \times \Theta$ we have

$$|E\left[u(\theta, \pi_0^{\Theta}, x(\pi_0^{\mathcal{M}}, \theta)) | \mathcal{P}^*\right] - E\left[u_N(\theta, \pi^{\Theta}, x_N(\pi^{\mathcal{M}}, m^N(\theta))) | |\mathcal{P}^*\right] < \varepsilon \tag{6.74}$$

with probability at least $1 - p$. Since the nonatomic game is informationally efficient, this immediately implies that for any (ε, ρ) the large finite game is informationally (ε, ρ)-efficient for N sufficiently large. □