Equilibrium Paths in Discounted Supergames ⋆

Kimmo Berg

Systems Analysis Laboratory, Aalto University School of Science and Technology, P.O. Box 11100, FI-00076 Aalto, Finland

Mitri Kitti ∗

Department of Economics, Aalto University School of Economics, P.O. Box 21240, FI-00076 Aalto, Finland

Abstract

We characterize subgame perfect pure strategy equilibrium paths in discounted supergames with perfect monitoring. It is shown that all the equilibrium paths are generated by fragments called elementary subpaths. When there are finitely many elementary subpaths, all the equilibrium paths are represented by a directed multigraph. Moreover, in that case the set of equilibrium payoffs is a graph directed self-affine set. The Hausdorff dimension of the payoff set is discussed.

Key words: repeated game, subgame perfect equilibrium, equilibrium path, payoff set, multigraph, fractal

JEL Classification: C72, C73

1 Introduction

Repeated games provide the most elementary setting for analyzing dynamic interactions. We consider the case where a stage game is repeated infinitely many times, players discount the future payoffs, observe perfectly each others' actions, and use pure strategies. These games have usually enormously rich sets of equilibrium strategies, which is generally thought to imply that the outcomes are hard to predict. Contrary to this conclusion, we show that all the outcome paths can be quite easily generated from a collection of subpaths.

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∗ Corresponding author.

Email addresses: kimmo.berg@tkk.fi (Kimmo Berg), mitri.kitti@hse.fi (Mitri Kitti).

Our approach relies on the characterization of subgame perfect equilibrium (SPE) payoffs by Abreu et al. (1986, 1990), hereafter APS. The APS theorem tells that SPE payoffs are a fixed-point of a particular iterated function system. The result holds for games with both imperfect and perfect monitoring, for the latter case see Cronshaw and Luenberger (1994). Cronshaw (1997) and Judd et al. (2003) point out that the APS characterization is similar to the Bellman equation in dynamic programming. To continue this analogue, our approach is similar to the Euler equation: we proceed to equilibrium paths from equilibrium payoffs.

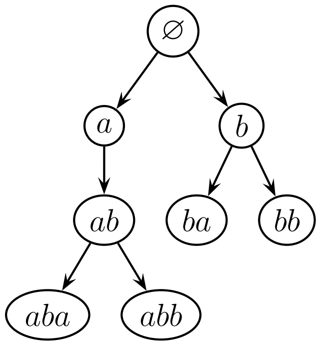
Our main result is that equilibrium paths are completely characterized by a collection of fragments or subpaths of them. Moreover, we show how these fragments, called elementary subpaths, can be found without knowledge on the whole set of equilibrium payoffs. The equilibrium paths are inherently related to payoffs: we shall observe that the payoff set is a sub-self-affine set in the sense of Käenmäki and Vilppolainen (2009), i.e., a particular fractal. Furthermore, when there are only finitely many elementary subpaths, the set is a graph-directed self-affine set. Consequently, it possible to analyze the Hausdorff dimension of the payoff set using tools developed for this kind of fractals.

The phenomenon that the set of equilibrium payoffs behaves in a rather complex manner is not completely new. Mailath et al. (2002) have analyzed pure strategy equilibria in repeated prisoners' dilemma using the approach of Abreu et al. (1986, 1990), and they have observed that there are discount factors for which maximum efficient payoff is not an equilibrium. We shall consider prisoners' dilemma as an example. Recently, Salonen and Vartiainen (2008) have demonstrated that the set of payoffs in a simple decision process can be highly complex, containing holes and caves. We offer a more comprehensive view to the complexities of payoffs in supergames: when the discount factors vary, the elementary subpaths change, which affects the properties of the iterated function system that generates the payoffs.

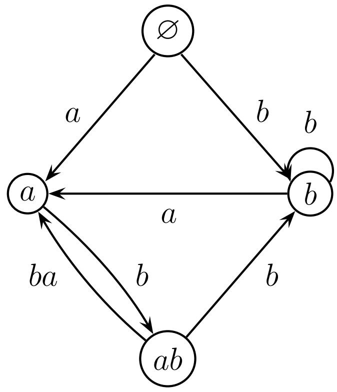
To clarify the approach of the paper let us consider a two-player game in which a and b are two action profiles. The common discount factor is δ. If we know that the SPE payoffs are $V^* \subset \mathbb{R}^2$, we can determine whether it is possible to play the subpath ab as a part of any equilibrium path. First, let us observe that there is a set of possible continuation payoffs for b, i.e., payoffs in V^* that can follow b on an equilibrium path. Let $C_b(V^*) \subseteq V^*$ denote this set. We shall later define the possible continuation payoffs more formally.

If any path starting with ab and followed by a payoff $v \in C_b(V^*)$ yields a payoff $(1-\delta)u(b) + \delta v \in C_a(V^*)$, where $u(b) \in \mathbb{R}^2$ is the vector of payoffs from action profile b, then any equilibrium path beginning with b can follow ab. By picking any initial action profile we can define this kind of fragments, or subpaths, which can belong to equilibrium paths. This leads to the set of subpaths that we call an elementary set. All the equilibrium paths can be generated by combining the subpaths of the elementary set.

When the elementary set consists of finitely many subpaths, then all equilibrium paths are represented by a directed multigraph. For example, if the elementary paths are *aba*, *abb*, *ba*, *bb*, then the equilibrium paths are obtained from the multidigraph in Figure 1.



(a) Tree



(b) Multidigraph

Figure 1. An example of elementary subpaths as a tree and a multidigraph presentation.

The symbol ∅ and the arcs starting from it denote that we can start with either playing a or b. Moreover, by playing b after a we get to node b after which we can either play b again or aa. Each arc on the graph corresponds to a subpath that is played after action profile indicated by the current node.

The multidigraph presentation can be used in creating all the possible equilibrium payoffs. Namely, each action profile corresponds to an affine mapping. For example, if δ is the common discount factor, a corresponds to a mapping $(1-\delta)u(a)+\delta v$ where v is the argument of the mapping. Consequently each arc corresponds to an affine mapping, too. Equilibrium payoffs are then generated by taking all the possible sequences of mappings that the graph allows. The resulting payoff set is a graph directed self-affine set. The computation of equilibrium payoffs in repeated games has been previously studied by Cronshaw (1997), and Judd et al. (2003). In these papers the payoff set is convexified by assuming correlated strategies. Our work thus contributes to the earlier works on the computation of equilibrium payoffs when the players use only pure strategies.

The paper is structured as follows. In Section 2, we restate the APS characterization of equilibrium payoffs. Equilibrium paths are considered in Section 3. In Section 4, we focus on finite subpaths and the computation of equilibrium subpaths. The properties of the equilibrium payoffs are analyzed in Section 5.

Conclusions are discussed in Section 6.

2 Subgame Perfect Equilibria

2.1 Discounted Supergames

We assume that there are n players, $N = \{1, \ldots, n\}$ denotes the set of players. The set of actions available for player i in the stage game is A_i. We assume that A_i, $i \in N$, are finite sets. The set of action profiles is denoted as $A = \times_i A_i$. Moreover, a_{-i} denotes the action profile of other players than player i, the corresponding set of action profiles is $A_{-i} = \times_{j \neq i} A_j$. Function $u : A \mapsto \mathbb{R}^n$ gives the vector of payoffs that the players receive in the stage game when a given action profile is played, i.e., if $a \in A$ is played, player i receives payoff $u_i(a)$.

In the supergame the stage game is infinitely repeated, and the players discount the future payoffs with discount factors δ_i, $i \in N$. We assume perfect monitoring: all players observe the action profile played at the end of each period. A history contains the path of action profiles that have previously been played in the game. The set of length k histories or paths is denoted as $A^k = \times_k A$. The empty path is $\varnothing$, i.e., $A^0 = \{\varnothing\}$. Infinitely long paths are denoted as A^∞. When referring to the set of paths beginning with a given action profile a we use $A^k(a)$ and $A^\infty(a)$ for length k paths and infinitely long paths, respectively. Moreover, $\mathcal{A}$ is the set of all paths, finite or infinite, and $\mathcal{A}(a)$ is the set of all paths that start with a, i.e., union of $A^k(a)$, $k = 1, 2, \ldots$ and $A^\infty(a)$.

A strategy for player i in the supergame is a sequence of mappings $\sigma_i^0, \sigma_i^1, \ldots$ where $\sigma_i^k : A^k \mapsto A_i$. The set of strategies for player i is Σ_i. The strategy profile consisting of $\sigma_1, \ldots, \sigma_n$ is denoted as σ. Given a strategy profile σ and a path p, the restriction of the strategy profile after p is is $\sigma|p$. The outcome path, simply path, that σ induces is $(a^0(\sigma), a^1(\sigma), \ldots) \in A^\infty$, where $a^k(\sigma) = \sigma^k(a^0(\sigma) \cdots a^{k-1}(\sigma))$ for all k.

The discounted average payoff for player i corresponding to strategy profile σ is

$$U_i(\sigma) = (1 - \delta_i) \sum_{k=0}^{\infty} \delta_i^k u_i(a^k(\sigma)). \tag{1}$$

Subgame perfection is defined in the usual way; σ is a subgame perfect equilibrium of the supergame if

$$U_i(\sigma|p) \geq U_i(\sigma_i', \sigma_{-i}|p) \;\forall i \in N,\; p \in A^k,\; k \geq 0, \text{ and } \sigma_i' \in \Sigma_i.$$

This paper focuses on SPE paths defined as below.

Definition 1 *$p \in A^\infty$ is a subgame perfect equilibrium path (SPEP) if there is a SPE strategy profile that induces p.*

2.2 Characterization of SPE Payoffs

Abreu et al. (1986, 1990) showed that the set of SPE payoffs is a fixed-point of a particular set-valued monotone operator. The result holds under both imperfect and perfect monitoring (Cronshaw and Luenberger, 1994). Kitti (2010) provides a generalization for a particular class of strategies in stochastic dynamic games with perfect monitoring. We restate the APS theorem under perfect monitoring for pure strategies.

For a compact set of payoffs $V \subset \mathbb{R}^n$, let us denote $v_i^-(V) = \min\{v_i : v \in V\}$, i.e., $v_i^-(V)$ is the minimal payoff for player i over V. A pair (a, v) of an action profile $a \in A$ and a continuation payoff $v \in V$ is admissible with respect to V if it satisfies the incentive compatibility constraint

$$(1-\delta_i)u_i(a) + \delta_i v_i \geq \max_{a_i \in A_i} \left[(1-\delta_i)u(a_i, a_{-i}) + \delta_i v_i^-(V)\right] \ \forall i \in N. \qquad (2)$$

This constraint says that it is better for player i to take action a_i and get the payoffs v_i than to deviate and then obtain $v_i^-(V)$.

Given a set of $V \subset \mathbb{R}^n$, we define the set of feasible action profiles, $F(V)$, to consist of action profiles for which there is $v \in V$ such that (a, v) is admissible. For $a \in F(V)$, we denote the set of possible continuation payoffs as $C_a(V)$, i.e., $v \in C_a(V)$ if (a, v) is admissible. Note that the vector of least payoffs that make (a, v) admissible can be found from the incentive compatibility condition. We let $\text{con}(a)$ denote this vector. It is the unique solution of the system of equations

$$(1-\delta_i)u(a) + \delta_i v_i = \max_{a_i \in A_i} \left[(1-\delta_i)u_i(a_i, a_{-i}) + \delta_i v_i^-(V^*)\right], \ i \in N.$$

Note that $C_a(V) = \{v \in V : v \geq \text{con}(a)\}$, where the inequality means that $v_i \geq \text{con}_i(a)$ for all $i \in N$.

We define an affine mapping $B_a : \mathbb{R}^n \mapsto \mathbb{R}^n$ corresponding to an action profile $a \in A$ by setting

$$B_a(v) = (I - T)u(a) + Tv,$$

where I is $n \times n$ identity matrix and T is a $n \times n$ diagonal matrix with discount factors $\delta_1, \ldots, \delta_n$ on the diagonal. Now we are ready to state the APS theorem.

Proposition 1 *Subgame perfect equilibrium payoffs are the unique largest compact set V^* for which*

$$V^* = \bigcup_{a \in F(V^*)} C_a(V^*) = \bigcup_{a \in F(V^*)} B_a(C_a(V^*)).$$

Proof. The result for the equality of V^* and the union of $B_a(C_a(V^*))$,$a \in A$, follows directly from the APS theorem according to which V^* is the largest set for which $V^* = B(V^*)$, where

$$B(V^*) = \bigcup_{(a,v)} \{B_a(v) : (a, v) \text{ admissible w.r.t. } V^*\}.$$

Let us now show the first equality. The inclusion that the right hand side is contained in V^* is obvious. Hence, we are left to show that $v \in V^*$ implies that v belongs also to the right hand side set. If this was not the case then there were no a such that (a, v) is admissible. This is in contradiction with v being SPE. $\square$

The operator B in the above proof is monotone in the set inclusion, i.e., $B(S^1) \subseteq B(S^2)$ when $S^1 \subseteq S^2$. The monotonicity implies that the iteration $W^{k+1} = B(W^k)$ converges monotonically to V^* when $V^* \subseteq W^0$. This idea has been further developed for computational purposes by Cronshaw (1997) and Judd et al. (2003) when a public correlation device is available and players have equal discount factors.

Proposition 1 tells that V^* is a fixed-point of the iterated function system defined by B_a, $a \in A$, and incentive compatibility constraints. In the particular case when $C_a(V^*) = V^*$ for all $a \in F(V^*)$, the set V^* is a self-affine fractal. Otherwise, it is not self-affine but it is contained in the self-affine set defined by the action profiles, i.e., it is a sub-self-affine set. We shall later return to the structure of the payoff set and describe how V^* is generated. In addition to denoting SPE payoffs as V^* we shall denote these payoffs corresponding to u and T as $V(u, T)$.

Note that it is possible that there are no subgame perfect equilibria. When the stage game has pure strategy equilibria, then it holds that $V^* \neq \emptyset$. However, even when there are no pure strategy Nash equilibria, the repeated game may have a non-empty set of subgame perfect equilibria.

There is one important observation to be made from APS characterization. It is based on the fact that a path p is a SPEP if and only if it is supported by the threat of reverting to the path giving $v_i^-(V^*)$ to the player i who deviated. The paths that yield the least equilibrium payoffs $v_i^-(V^*)$, $i \in N$, form an extremal penal code (Abreu, 1986, 1988). Notice that the payoffs $v_i^-(V^*)$, $i \in N$, are exactly what appear on the right hand side of players' incentive compatibility constraints.

3 Equilibrium Paths and Subpaths

In this section we analyze the paths that yield payoffs in the set V^* that is the set of SPE payoffs. For $p \in \mathcal{A}$, we p_j is that starts from the element $j+1$, and p^k is the path of first k elements of p. More specifically, when $p = a^0 a^1 \cdots$, we have $p_1 = a^1 a^2 \cdots$, $p^k = a^0 \cdots a^{k-1}$, and $p_j^k = a^j \cdots a^{j+k-1}$.

The length of path p is denoted as $|p|$, $i(p)$ is the initial and $f(p)$ the final element of p. If p is infinitely long, in brief an infinite subpath, then $f(p) = \varnothing$. If p and p' are two paths then pp' is the path obtained by juxtaposing the terms of p and p'. We shall focus on fragments of SPEPs. These fragments will be called equilibrium subpaths.

Definition 2 *A path $p' \in \mathcal{A}(a)$ is a SPE subpath if there is a SPEP $p \in A^\infty(a)$ such that $p^{|p'|} = p'$.*

Let us consider the set of tail payoffs that are possible after an action profile a when it begins a path $p \in \mathcal{A}(a)$ that may be finitely or infinitely long. We also assume that p is a SPE subpath. If p is infinite then it is an equilibrium path itself. First, we can observe that a should be followed by a payoff that belongs to $C_a(V^*)$. As a is followed by p_1, we need consider what are the payoffs that $i(p_1)$ generates from the set of tail payoffs that are possible for $i(p_1)$ when it is followed by p_2.

Let $W(p)$ denote the set of continuation payoffs that are possible for a after $p \in \mathcal{A}(a)$. Then it holds that this set satisfies the recursion

$$W(p) = C_a(V^*) \cap B_{i(p_1)}(W(p_1)).$$

Namely, the continuation payoff for a should belong to $C_a(V^*)$ and it should be generated by $i(p_1)$ from $W(p_1)$. Note that for $W(p)$ we need $W(p_1)$ for which we need $W(p_2)$ and so on. In particular, if $|p| = \infty$, the recursion is infinite and $W(p)$ becomes a singleton. To complete the definition of $W(p)$ we set $W(\varnothing) = V^*$ and $B_\varnothing = I$. This is needed because p_1 is an empty path when $|p| = 1$. The following example clarifies the construction of $W(p)$.

Example 1 *Let us assume that there are two action profiles a and b, and let us consider the subpath abba. To find out $W(p)$ we start with the final element of the path, i.e., from a. Let us take $C_a(V^*)$. If $v \in C_a(V^*)$ is followed after a, the payoff that is obtained when taking first b is $(I-T)u(b)+Tv$, i.e., $B_b v$. More generally the set of payoffs that are possible after ba are $B_b(C_a(V^*))$.*

We can now consider the payoffs that are possible after bba. This set is simply $W(p_1) = B_b(B_b(C_a(V^*)))$. Eventually we obtain $W(p)$.

The following observations on $W(p)$ form a basis for the definition of elementary subpaths.

Remark 1 *Let $p \in \mathcal{A}(a)$ for $a \in A$.*

i) *p is a SPE subpath if and only if $W(p) \neq \emptyset$.*

ii) *If $W(p_1) \neq \emptyset$ and*

$$B_{i(p_1)}(W(p_1)) \subseteq C_a(V^*), \tag{3}$$

then pp' is a SPE subpath whenever $f(p)p'$ is a SPE subpath.

The second observation is particularly important. It says that any SPE subpath that follows from the final element of p constitutes another SPE subpath when it is juxtaposed with p. For example if abc is a SPE subpath such that $W(bc) \neq \emptyset$ and $B_b(W(bc)) \subseteq C_a(V^*)$, then ab can be followed by any SPE subpath that begins with c. Note also that, if $V^* \subseteq C_a(V^*)$, i.e., $V^* = C_a(V^*)$, then a can be followed by any SPE subpath.

The subpaths that satisfy (3) play a crucial role in the rest of the paper. In particular, they give us elementary subpaths.

Definition 3 *If $p \in \mathcal{A}(a)$ satisfies $W(p_1) \neq \emptyset$, condition (3), and there is no $k < |p|$, such that p^k satisfies these conditions, then p is an elementary subpath and we denote $p \in P^{|p|}(a)$.*

Note that we allow for infinitely long sequences; if $p \in P^\infty(a)$, then $C_a(V^*) \setminus B_{i(p_1)}(W(p_1^k)) \neq \emptyset$ for all k. The requirement that no restriction to any first k elements satisfies (3) means that P^k sets do not contain paths that already belong to other P^j sets. For example, if abc is an elementary subpath, then $abcd$ cannot be an elementary subpath even though it may satisfy (3).

The following result tells that all the SPEPs are characterized by the elementary subpaths, i.e., sets P^k and P^∞.

Proposition 2 *A path $p \in A^\infty(a)$ is a SPEP if and only if for all $j \in \mathbb{N}$ either $p_j^k \in P^k(i(p_j^k))$ for some k or $p_j \in P^\infty(i(p_j))$.*

Proof. By the construction of P^k's, a SPEP path p satisfies one of the two conditions of the proposition.

Let us assume that for all j either for all j we have $p_j^k \in P^k(i(p_j^k))$ for some k or $p_j \in P^\infty(i(p_j))$. In that case, for any j the payoff to player i is at least $v_i^-(V^*)$ when the players choose action profiles such that they stay on the subpath p_j^k or p_j. We first argue that in the case when there is k such that $p_j^k \in P^k(i(p_j^k))$, the threat of reverting to the path that yields $v_i^-(V^*)$ to the deviator i, keeps the players on path p. In that case it does not matter for players at stage j what happens after k periods as long as the continuation payoff after these periods is in V^* and the penalty from deviations is $v_i^-(V^*)$. On the other hand, if $p_j \in P^\infty(i(p_j))$ then the players do not have any incentive to deviate either. This means that the path p is supported by extremal penal code, i.e., there is an SPE strategy that yields p as an outcome. □

Example 2 *Let us assume that there are four action profiles; $A = \{a, b, c, d\}$. Moreover, let the sets $P^k(a)$, $k = 1, 2$, $a \in A$, be as in the table below. Now, $aa \in P^2(a)$ means that on any equilibrium path a should be followed by another a after which it does not matter what comes next as long as it is an equilibrium path. However, since only action profile that can follow a is a, we observe that after the first a on an equilibrium path, the rest of the action profiles are also a's. On the other hand, b can be followed by two action profiles; ba and bc, after which any equilibrium path starting with a or c, respectively, is possible. For the action profile c the situation is symmetric to that of b. Finally, since $d \in P^1(d)$, it can be followed by any equilibrium path. Now, all equilibrium paths can be presented as $d^{\mathbb{N}}b^{0,1}(cb)^{\mathbb{N}}c^{0,1}a^\infty$, where $b^{0,1}$ means b is either played once or is not played at all. Moreover, $d^{\mathbb{N}}$ means that d can be repeated arbitrarily many times, here $\mathbb{N} = \{0, 1, \ldots\}$, and a^∞ means that a is repeated infinitely many times.*

Table 1
An example of sets $P^1(a)$ and $P^2(a)$.

	a	b	c	d
P^1	$\emptyset$	$\emptyset$	$\emptyset$	$\{d\}$
P^2	$\{aa\}$	$\{ba, bc\}$	$\{ca, cb\}$	$\emptyset$

The sets P^k and P^∞ tell us what are the subpaths that can follow an initial action. The result of Proposition 2 says that for each action profile a on the equilibrium path there is a subpath belonging to $P^k(a)$ for some k or $P^\infty(a)$. This means that equilibrium paths follow a particular syntax defined by the elementary subpaths. In the rest of the paper we shall focus on the collections of elementary subpaths. These collections are called elementary sets. Particular attention will be paid to finite elementary sets and finite subsets of them.

Definition 4 *The collection of sets $P^k(a)$, $k = 1, 2, \ldots$, and $P^\infty(a)$, $a \in A$ is is an elementary set corresponding to the infinitely repeated game with payoffs u and discount factors determined by T. This collection is denoted as $S(u, T)$. Moreover, $S^k(u, T)$ contains $P^j(a)$, $j = 1, \ldots, k$, $a \in A$.*

Note that $S^k(u, T)$ contains finitely many subpaths because P^k are finite for all k. Another observation on finiteness of elementary subpaths is an immediate consequence of contractivity of B_a, $a \in A$. Namely, it follows that if $p \in \mathcal{A}(a)$

and $v(p_1) > \mathrm{con}(a)$, then there is k such that p^k satisfies (3). If p_1 is an empty path we set $v_i(p_1) = v_i^-(V^*)$ for all $i \in I$. More generally we have the following result.

Proposition 3 *For any $\varepsilon > 0$ there is k such that $p \in A^\infty(a)$, $a \in A$, $v(p_1) \geq$ con$(a) + \varepsilon$, imply that $p_j^l \in P^l(i(p_j))$ for $l \leq k$.*

Proof. Because A is finite and B_a, $a \in A$, are contractions, for any $\rho > 0$ there is k such that the diameter of the set that is obtained by taking the image of V^* under a sequence $B_{a^0}, \ldots, B_{a^{k-1}}$, $a^j \in A$ for all j, has the diameter less than ρ. In particular, the diameter of the set $B_{i(p_1)}(W(p_1^k))$ is less than ρ for any p. Now, ρ can be chosen such that $B_{i(p_1)}(W(p_1^k)) \subseteq \{v : v(p_1) > \mathrm{con}(a) + \varepsilon\}$, which concludes the proof. □

This result means that if there is an equilibrium path $p \in \mathcal{A}(a)$ for which (3) fails to hold for all p^k, then $v_i(p_1) = \mathrm{con}_i(a)$ for some i. The opposite is not true, i.e., we may have $v_i(p_1) = \mathrm{con}_i(a)$ for some $i \in N$ at the same time when (3) holds.

Let us now consider the comparative statics of $S(u, T)$ for T. Let T^1 and T^2 be two matrices corresponding to two different sets of discount factors. We denote $T^1 \ll T^2$ if the discount factors corresponding to T^2 (diagonal elements) are at least those of T^1. With a slight abuse of notation, we denote $p \in S(u, T)$ when there is a set $P^k(a)$ or $P^\infty(a)$, $i(p) = a$, such that p belongs to this set of subpaths.

Proposition 4 *If $T^1 \ll T^2$ then $p \in S(u, T^1)$ implies that there is $k \leq |p|$ such that $p^k \in S(u, T^2)$.*

In the following $P^k(a; u, T^i)$ denotes the set of length k elementary subpaths corresponding to T^i, $C_a(V(u, T^i); u, T^i)$ is the set of continuation payoffs, $v(p_k; u, T^i)$ is the payoff corresponding to p_k, and $W(p_k; u, T^i)$ is the set of possible continuations after p_k, when u and T^i are given.

Proof. As discount factors increase the set of equilibrium strategies enlarges. Consequently, if we take a path $p \in P^l(a; u, T^1)$ and a continuation payoff $v(f(p); u, T^1) \in C_{f(p)}(V(u, T^1); u, T^1)$, then $v(f(p); T^2) \in C_{f(p)}(V(u, T^2); u, T^2)$. By induction argument, $v(p_k; T^1) \in W(p_k; u, T^1)$ implies that $v(p_k; u, T^2) \in W(p_k; u, T^2)$ for all $k \leq l$. Hence, if $v(p_1; u, T^1) \in C_a(V(u, T^1); u, T^1)$, i.e., (3) holds, we also have $v(p_1; u, T^2) \in C_a(V(u, T^2); u, T^2)$. This means that either $p \in P^l(a; u, T^1)$ or there is $k \leq l$ such that $p^k \in P^k(a)$. If $p \in P^\infty(a; u, T^1)$ then p is a SPEP for T^1, and therefore it is a SPEP also for T^2. Again, either $p \in P^\infty(a; u, T^2)$ or there is k such that $p^k \in P^k(a; u, T^2)$. □

When the discount factor increases, all the subpaths that satisfy (3) still satisfy this condition. It may, however, happen that the number of elementary subpaths decreases and their maximum length decreases. For example, if $abcd, abdc \in S(u, T^1)$ it may happen that $ab \in S(u, T^2)$ for $T^2 \gg T^1$, i.e., corresponding to two elementary subpaths starting with ab there is only one when discount factors increase.

4 Finite Elementary Sets: Multidigraph Presentation

The main result of this section is that all the SPEPs are represented by a multidigraph when $S(u, T)$ contains finitely many subpaths. Even if the elementary set is not finite, the graph presentation is possible for the paths generated by $S^k(u, T)$, $k \geq 1$.

The idea in the construction of a multidigraph is based on first presenting all the elementary subpaths as a tree, then we transform this tree into a graph, and finally simplify the graph into a multidigraph. In this process the nodes and arcs are given labels. Equilibrium paths can be created from the multidigraph by joining different arcs. Consequently, the paths are determined by the labels of the arcs. Later we shall observe that the labels of arcs can be utilized in constructing payoff sets. The procedure for constructing the multidigraph is described below.

(1) Form a tree of the finite elementary paths. The root node is the empty history $\varnothing$. For example, let us assume that the elementary subpaths are aba, abb, ba, and bb. The tree presentation is given in Figure 1.

(2) Form a graph from the tree. Each node in the tree corresponds to a node in the graph. Form the arcs between the nodes by going through them and determine the destinations for each one.

   (a) The destinations of an inner node in the tree, i.e., node with children, are its children. Set an arc to each destination node. For example, in the previous example a-node has a destination ab.

   (b) The destinations of a leaf node, i.e., node with no children, which is connected to the root node $\varnothing$ are all the child nodes of $\varnothing$.

   (c) For the other leaf nodes, find smallest $k \geq 1$ such that p_k is found in the tree. For example, corresponding to the leaf aba in the tree we first search for ba in the tree. Since it is found we set the node ba as a destination and insert the corresponding arc.

(3) Forming a multidigraph. Simplify the graph obtained in previous step by removing the nodes that have only one destination and are not children of $\varnothing$. For example, let us assume that there are two arcs pointing to c, from a and b, and c has one destination to d. Because there is only one destination from c we remove node c and redirect the incoming arcs to the destination d. So we have arcs from a to d and b to d. We label both

these arcs as cd. The result of the simplification is a directed graph, where there may be multiple arcs between the nodes, i.e., multidigraph.

(4) Insert arcs and nodes for infinitely long subpaths. For each infinitely long elementary subpath find largest k such that p^k is a node. Insert an arc with label p^k from this node to a dummy node corresponding to the path.

As a consequence of the above construction we get the result stated below.

Proposition 5 *When $S(u,T)$ contains finitely many subpaths, then all SPEPs are represented by a multidigraph.*

An immediate corollary of the result is that when we take $S^k(u,T)$, then the multidigraph presentation is possible.

Corollary 1 *The SPEPs given by $S^k(u,T)$ can be presented as a multidigraph.*

In some cases the infinitely long elementary subpaths follow a particular syntax. In the following example we demonstrate this phenomenon. In that case it is possible to associate a smaller number of dummy nodes to the graph than in the previous construction.

4.1 Example of Constructing a Multidigraph

We assume that there are four action profiles: a, b, c, and d. The finite elementary subpaths are given by sets in Table 1 and $P^4(b) = \{bdca\}$, $P^4(c) = \{cdba\}$, the rest of $P^k(a)$, $a \in A$, being empty. These elementary subpaths are obtained for the prisoners' dilemma game. We show how this is done in the following section. If a is played then only a may follow, and if d is played in the beginning, then any action profile may follow. If b is played, then a may follow after which a is repeated, or dca may follow and also after this subpath a is repeated.

The tree of elementary subpaths is presented in Figure 2. The destinations of leaf nodes are indicated next to them. After forming a graph from this tree (Step 2) and simplifying it, we get the directed multigraph composed of solid arrows in Figure 2. The subpaths that are played after an action profile to get to another node are shown next to the arc if this subpath is different from the node to which the arc is directed.

We can see that there are three subpaths a, bc, and d that may be infinitely repeated. These correspond to the cycles in the multidigraph. The SPE path may move from d to either bc or a, and from bc to a, but a is absorbing, i.e., once it has been reached the play will never leave the node. There are two kind of subpaths that can be played only finitely many times. Those that can be played at the beginning of the game and those that can be played in the middle of the game. For example, we may have b in ba^∞ as an example of the first kind, and bdc in $d^{\mathbb{N}}bdca^\infty$ as a subpath of the second kind. Indeed, these will be equilibrium paths in the prisoners' dilemma game as will be seen. We can also see that there are two transitions from both nodes b and c to node a, i.e., paths ba and $bcda$.

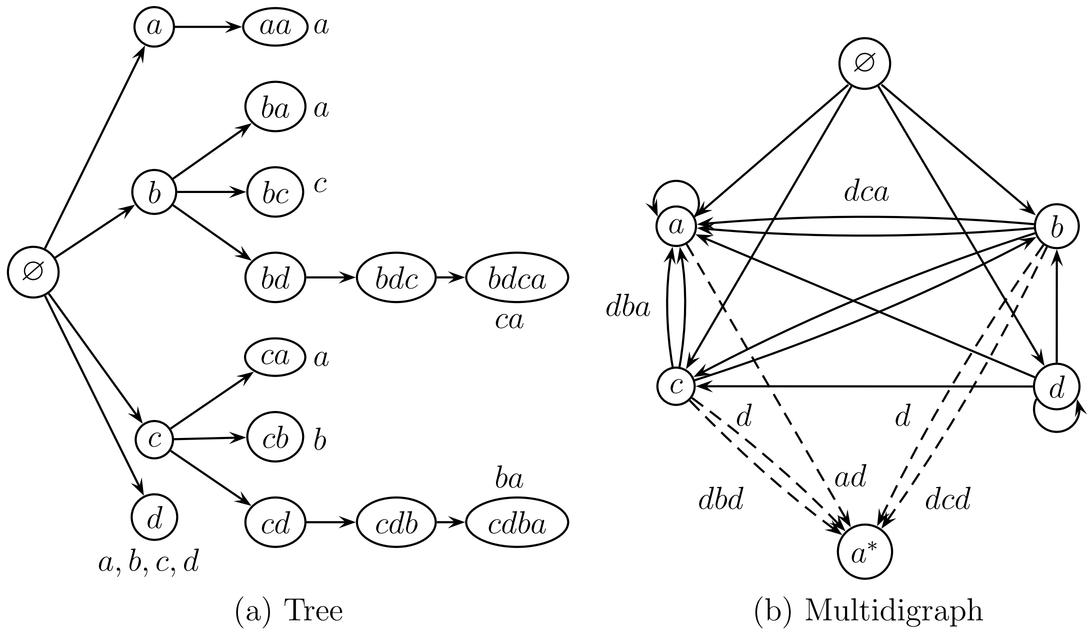


Figure 2. Tree of finite elementary subpaths and a multidigraph presentation of all elementary subpaths.

To get all the SPEPs of the game, we add nodes and arcs corresponding to infinitely long elementary subpaths to the multidigraph. Let us assume that these paths are

$$\{ada^\infty, bda^\infty, cda^\infty, bdcda^\infty, cdbda^\infty\}.$$

We shall see that these will be the infinitely long elementary subpaths in prisoners' dilemma for $\delta = 1/2$. We need another node to distinguish whether d is played after a, b, or c or not. For example, if ad is played then a^∞ must follow and ad cannot be played any more. We denote this extra node as a^*, and by adding the new transitions we get the multidigraph of Figure 2 in which the dashed arrow are also included.

4.2 Computation of Finite Elementary Subpaths

Let us consider a subpath abc. The vector of least payoffs $\mathrm{con}(ab)$ that the players should get after ab to make the first element a incentive compatible are found by solving a system $(I-T)u(b)+Tv = \text{con}(a)$. Given that $\text{con}(ab)$ is known we can now find the least payoff that is required after abc to make a incentive compatible as the first action profile. This continuation payoff $\text{con}(abc)$ solves $(I-T)u(c)+T\text{con}(abc) = \text{con}(ab)$. If it happens that $\text{con}(abc) \leq \text{con}(c)$, then any equilibrium path starting from c is a possible continuation for abc. This exactly the same thing that (3) says.

In general we can define $\text{con}(p)$ for any $p \in A^k$, $k \geq 2$, as above. When $\text{con}(p^{k-1})$ is known and $p = p^{k-1}a^k$, we set

$$\text{con}(p) = T^{-1}\left[\text{con}(p^{k-1}) - (I-T)u(a^k)\right].$$

Now $\text{con}(p)$ is simply the continuation payoff that is required after $f(p)$ to make the first action profile of p incentive compatible. The following observations are immediate.

Remark 2 *Let $p \in A^k$ and $\bar{v}_i = \max\{v_i : v \in V^*\}$, $i \in N$.*

i) *Condition (3) holds for $p \in A^k$ with $f(p) = a$ if and only if $W(p) \neq \emptyset$ and $con(p) \leq con(a)$.*
ii) *If $con_i(p) > \bar{v}_i$ for $p \in A^k$ and some $i \in N$, then p is not an elementary subpath.*

Notice that to detect whether a subpath is elementary or not does not require knowing the whole set of equilibrium payoffs. The above properties can be efficiently utilized in the computation of the elementary subpaths.

The algorithm for finding the elementary subpaths is described below.

1. For all $a \in A$ include $a \in P^1(a)$ if $\text{con}_i(a) \leq v_i^-(V^*)$ for all $i \in N$. If, $v_i^-(V^*) \leq \text{con}_i(a) \leq \bar{v}_i$ for all $i \in N$, and the first inequality is strict for some $i \in N$, then include a in $P_*^1(a)$. Set $k = 2$, and go to Step 2.
2. For each $a \in A$, find $\text{con}(pb)$ for all $b \in A$ and $p \in P_*^{k-1}(a)$.
   a) If $\text{con}(pb) \leq \text{con}(b)$ and $P^j(b) \neq \emptyset$ for some $j \leq k$, include pb in $P^k(a)$.
   b) Otherwise, if $\text{con}_i(pb) \leq \bar{v}_i$ for all $i \in N$ and $[\cup_{j<k}P^j(b)] \cup P_*^{k-1}(b) \neq \emptyset$, include pb in $P_*^k(a)$.

   If $P_*^{k+1}(a) = \emptyset$ for all $a \in A$ stop. Otherwise, increase k by one and repeat Step 2.

The set $P_*^k(a)$ contains the subpaths that are possibly part of elementary subpaths. The test in step 2.b) tells whether it is possible that pb is part of an elementary subpath. First, the required continuations should not exceed the upper bounds $\bar{v}_i$, $i \in N$. Second, either there is a shorter elementary subpath starting with b or there is possibly some elementary subpath starting with b, i.e., subpath in $P_*^{k-1}(b)$.

Remark 3 *If there is k such that $P_*^k(a) = \emptyset$ for all $a \in A$, then $S(u,T)$ contains finitely many subpaths.*

4.3 Example of Finding Elementary Subpaths

We consider prisoners' dilemma game with a common discount factor $\delta = 1/2$ and the stage-game payoffs as below.

	L	R
T	$3,3$	$0,4$
B	$4,0$	$1,1$

We denote the action profiles from left to right and top to bottom as a,b,c, and d; for example (T,R) is b. The penal path is d^∞. The punishment payoffs are $v_i^-(V^*) = 1$, $i = 1,2$.

The aim is to find the elementary set for this game. We classify the finite paths into elementary and non-elementary sets, and increase the path length until the neutral set becomes empty. For example, $\text{con}(d) = (1,1)$, and d is an elementary subpath since $\text{con}_i(d) \le v_i^-(V^*)$, $i = 1,2$. The following table gives the payoff requirements for one and two length paths. The elementary subpaths are indicated by $+$, the non-elementary subpaths by $-$, and those which belong to $P_*^k(a)$, $k = 1,2$, $a \in A$, by $*$. Since d is an elementary path, we do not need to examine paths da, db, dc, dd.

Table 2
Finding elementary subpaths with $|p| \le 2$ in prisoners' dilemma game.

path	con(path)	path	con(path)	path	con(path)
a	$(2,2)^*$	b	$(2,1)^*$	c	$(1,2)^*$
aa	$(1,1)^+$	ba	$(1,-1)^+$	ca	$(-1,1)^+$
ab	$(4,0)^-$	bb	$(4,-2)^-$	cb	$(2,0)^+$
ac	$(0,4)^-$	bc	$(0,2)^+$	cc	$(-2,4)^-$
ad	$(3,3)^*$	bd	$(3,1)^*$	cd	$(1,3)^*$

From the table, we can conclude that the nonempty sets $P^k(a)$, $k \le 2$, are as given in Table 1. What are left to search for are the paths beginning with ad, bd, and cd. We can immediately observe that ad is incentive compatible only when it is followed by an infinite repetition of a, i.e., $P^\infty(a) = \{ada^\infty\}$, since no other action profiles give the required payoff $(3,3)$. Moreover, due to symmetry, we only need to check paths beginning with b or c.

Let us consider the three and four length paths beginning with *cd*. For example, *cda* belongs to $P^3_*(c)$ because $\mathrm{con}_i(cda) \leq 3$, $i \in N$, and $[\cup_{j\leq 2} P^j(a)] \cup P^2_*(a) = \{aa, ad\}$.

Table 3
Finding elementary subpaths with $3 \leq |p| \leq 4$ in prisoners' dilemma game.

path	con(path)	path	con(path)	path	con(path)
cda	$(-1, 3)^*$	*cdaa*	$(-5, 3)^*$	*cdba*	$(1, 1)^+$
cdb	$(2, 2)^*$	*cdab*	$(-2, 2)^{*,-}$	*cdbb*	$(4, 0)^-$
cdc	$(-2, 6)^-$	*cdac*	$(-6, 6)^-$	*cdbc*	$(0, 4)^-$
cdd	$(1, 5)^-$	*cdad*	$(-3, 5)^-$	*cdbd*	$(3, 3)^*$

Now, we can see that the only possible paths starting with *cd* are starting with *cda* and *cdb*. From length four paths, we can observe that $P^4(c) = \{cdba\}$ and $P^\infty(c) = \{cdbda^\infty, cda^\infty\}$. The only continuation to *cdaa* is *aa*, since the only elementary subpaths starting with *a* are *aa* and *ad*, and *ad* gives lower payoff than the required $(-5, 3)$. Notice that *cdab* is ruled out because *ab* cannot be on an equilibrium path, since it is a non-elementary path. The path *cdbb* could be ruled out by similar reasoning. Hence, there are no longer paths to be searched for and we have found the elementary subpaths for the game.

5 Properties of the Payoff Set

In this section we discuss the Hausdorff dimension of SPE payoffs. Intuitively, the dimension gives a measure for the complexity of the payoff set. As mentioned previously, the set of equilibrium payoffs is a sub-self-affine set. This means that V^* belongs to the self-affine set determined by B_a, $a \in A$, i.e., $V^* \subseteq W$, where W satisfies

$$W = \bigcup_{a\in A} B_a(W).$$

The difference between V^* and W is that the set of contractions that define V^* depends on V^* itself, and these contractions do not map the whole set V^* but only subsets that satisfy incentive compatibility conditions. This breaks the self-affinity in the payoff set. However, when $S(u, T)$ contains finitely many subpaths the payoff set is a graph directed self-affine set in the sense of Mauldin and Williams (1988).

A graph directed self-affine set is an attractor of a graph directed iterated function system. Each arc on the multidigraph corresponds to an affine contraction mapping. For a supergame these affine contractions are given by the labels of the arcs in the multigraph presentation of elementary subpaths. For example, if an arc has a label *acdc*, the corresponding affine mapping is the composition $B_aB_cB_dB_c$. This is a contraction because discount factors are less than one, i.e., T is a contraction. For the nodes corresponding to infinitely long elementary subpaths the corresponding mappings can be associated with the values that these subpath yield.

Proposition 6 *When $S(u,T)$ contains finitely many subpaths, $V(u,T)$ is a graph-directed self-affine set.*

The result is a consequence of Proposition 5. As sketched above, it follows by associating an affine mapping to each arc of the multigraph presentation. Let E_{qr} denote the list of arcs from q to r, and M the list of nodes in the multigraph presentation. For example, in the prisoners' dilemma game $E_{ca} = \{a, cba\}$. The invariant sets corresponding to the graph-directed construction satisfy

$$V_q = \bigcup_{r\in M} \bigcup_{p\in E_{qr}} B_p(V_r), \text{ for all } q \in M,$$

where B_p denotes the affine mapping corresponding to the arc p. Furthermore, we have $V(u,T) = \cup\{V_r : r \in M\}$.

To demonstrate how the infinitely long elementary subpaths are treated we may consider the arc cd from c to a^* in prisoners' dilemma. Corresponding to a^* we have the payoff $(3,3)$ and we set $B_{cd}v = B_cB_d(3,3)$ for all $v \in \mathbb{R}^2$.

In general, it is hard to say much about the exact dimension of the graph-directed self-affine sets. Upper and lower estimates of the dimension have been discussed by Edgar and Golds (1999). The following proposition is obtained for small discount factors as a consequence of a recent result by Käenmäki and Vilppolainen (2009). Topological pressure needed in the proposition is a particular function that is defined by SPEPs. Let us assume that the players are indexed according to the order of discount factors such that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. The singular value function of T^j, i.e., j-times product of T, is then

$$\phi^t(T^j) = \begin{cases} (\delta_1\delta_2\cdots\delta_{m-1})^j\delta_m^{j(t-m+1)}, & 0 \leq t < n, \\ (\delta_1\delta_2\cdots\delta_n)^{jt/n}, & t \geq n, \end{cases}$$

where m is the integer such that $m-1 \leq t \leq m$. Let K denote the set of all SPEPs, $K_j = \{p_j \in A^j : p \in K\}$, and $\#K_j$ the number of elements in K_j.

When $S(u, T)$ has finitely many subpaths, K and K_j, $j \geq 1$, are determined by the multidigraph presentation. The topological pressure (Falconer, 1995, Käenmäki and Vilppolainen, 2009) takes the form

$$P(t) = \lim_{j \to \infty} \frac{\log\left[\phi^t(T^j)(\#K_j)\right]}{j}.$$

In the following $s(u, T)$ denotes the zero of the topological pressure for given u and T.

Proposition 7 *Let us assume that $\delta_i < 1/2$ for all $i \in N$. Then the Hausdorff dimension of $V(u, T)$ is $\min\{n, s(u, T)\}$ for Lebesgue-almost all payoff functions u for which $V(u, T) \neq \emptyset$.*

Proof. The result follows from Theorem 5.2 in Käenmäki and Vilppolainen (2009). For the assumptions of the theorem we need three properties. First, T should satisfy $\|T\| < 1/2$, where $\|T\|$ is the largest singular value of T, i.e., the square root of largest eigenvalue of $T \times T$. It easy to observe that $\|T\| = \max_i \delta_i$. Consequently, $\|T\| < 1/2$ when $\delta_i < 1/2$ for all $i \in N$. Second, p_1 should be a SPEP whenever p is a SPEP. This is obviously the case. The third property that we need is that the set of SPEPs should be compact in the topology of the metric defined by the distance

$$|p - r| = \begin{cases} \alpha^{\min\{k-1: p_k \neq r_k\}}, & p \neq r, \\ 0, & p = r, \end{cases}$$

where $\alpha \in (0, 1)$.

To obtain compactness we first associate a common element to all action profiles that yield the same payoff. For example, if $u(a) = u(b)$ for $a, b \in A$, $a \neq b$, we can simply replace every b on all paths with a. Now take a sequence of SPEPs $p(j)$, $j = 1, 2, \ldots,$. The sequence of payoffs corresponding to these paths has a convergent subsequence. This is because V^* is a compact set when non-empty. Let $p(j_k)$ denote the subsequence of SPEPs corresponding to this subsequence of payoffs, and let p denote a path corresponding to the limit. Because $\delta_i < 1/2$ it holds that $B_a(V^*) \cap B_b(V^*) = \emptyset$ when $a, b \in A$ and $u(a) \neq u(b)$. This together with the assumption that all the action profiles on equilibrium paths have different payoffs implies that the limit path is unique. Moreover, the no-overlapping property guarantees that only the final elements of $p(j_k)$ can differ from p, and when k is increased the threshold for k above which the elements are different increases. This means that $|p(j_k) - p|$ goes to zero as k increases, which proves the compactness. □

In general the Hausdorff dimension increases as the affine mappings become less contractive, i.e., discount factors increase. This observation follows directly from the definition of topological pressure. However, it is well known for self-affine sets that there can be exceptional points, i.e., payoff functions u, for which the zero of topological pressure does not give the dimension but only an upper bound, see, e.g., Falconer and Miao (2008). Hence, it may happen that as discount factors increase the Hausdorff dimension drops suddenly.

5.2 Examples

To illustrate the possible payoff sets that can be obtained from multidigraph presentations we consider two examples. The prisoners' dilemma game and the following game, called Sierpinski game. The payoffs are as given below and $\delta = 1/2$. We also denote $a = (T, L)$, $b = (C, M)$, and $c = (B, R)$.

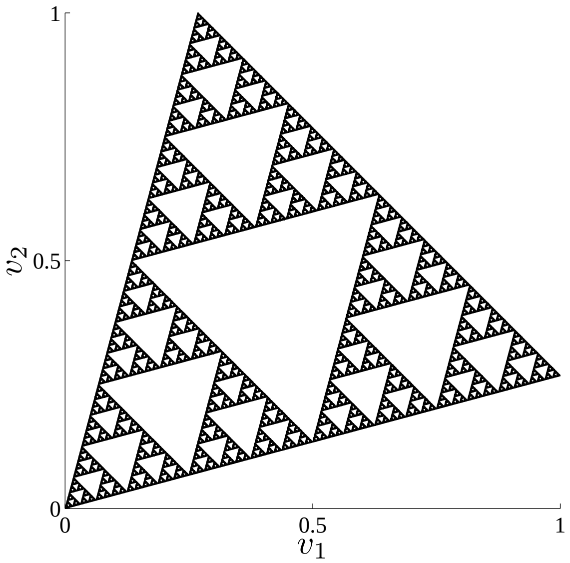
	L	M	R
T	$2-\sqrt{3}, 1$	$-1, -1$	$-1, -1$
C	$-1, -1$	$1, 2-\sqrt{3}$	$-1, -1$
B	$-1, -1$	$-1, -1$	$0, 0$

In this game there are three pure strategy Nash equilibria, which are the corner points of V^*. The set set V^*, illustrated in Figure 3, is one of the most famous fractals—Sierpinski triangle. It is well known that the Hausdorff dimension of the set is $\ln 3/\ln 2 \approx 1.585$. This value tells that the set does not quite fill the two dimensional space but on the other hand it is more complex than one dimensional sets.

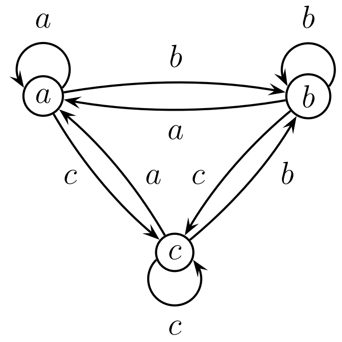
The set V^* is generated by arbitrary paths of combinations of the three Nash equilibria. The payoff of any infinite path that is a combination of these three points correspond to a point in the Sierpinski triangle. The multidigraph for the elementary subpaths consists of these three action profiles and all transitions between them (and the loops to itself), see Figure 3. Here the dummy node $\varnothing$ is omitted as redundant.

We can make interesting observation on the effect of changing the discount factor in the Sierpinski game. In addition to affecting the elementary set as shown in Proposition 4, discounting defines the scale of the payoff set. When the discount factor is increased a little and the elementary set does not change, then only the distance between the points in the SPE payoff set is decreased.

In Sierpinski game the payoff set fills the triangle defined by the three Nash equilibria when $\delta > 2/3$, i.e., the Hausdorff dimension becomes two. This happens even if the set of elementary subpaths remains the same when discount factor increases. For example, we can replace minus ones by a small enough



(a) SPE payoffs—Sierpinski triangle



(b) Multidigraph

Figure 3. Sierpinski triangle as the SPE payoff set and the multidigraph presentation of SPEPs.

number to guarantee that there will be no more equilibrium paths when δ increases. This gives an important insight into the folk theorem for discounted supergames (see, e.g., Fudenberg and Maskin, 1986): one reason for the fact that any feasible payoff above min-max levels can be achieved as an SPE outcome is that the payoffs are less contracted under B_a, $a \in A$, when the discount factors increase. Moreover, the set of payoffs may enlarge even when the set of equilibrium paths and hence strategies remains the same.

Finally, let us examine the payoff set in the repeated prisoner's dilemma game. The payoff sets of variations of this supergame have previously been studied, e.g., by Sorin (1986), Stahl (1991), and Mailath et al. (2002). The payoff sets for common discount factors $\delta = 0.5$ (left) and $\delta = 0.58$ (right) are illustrated in figure 5.2. The sets consists of similar parts, which shows the fractal nature. The sets are constructed by generating finite paths using the multidigraph presentations and combining to them the cycles starting from the final actions of the paths. For $\delta = 0.5$ the set is rather sparse and its Hausdorff dimension is zero. The payoff requirements for the first and second columns are presented as dashed and solid lines, respectively. We can also see that there are points on these lines, and these correspond to the paths in P^{∞}, like ada^{∞}, bda^{∞} and cda^{∞}. This is the role of the infinitely long elementary paths, i.e., some part of the path gives exactly the minimum payoff requirement. For $\delta = 0.58$ the payoff set has much more structure and its Hausdorff dimension is approximately 1.4.

The complexity of SPEPs and the Hausdorff dimension comes down to the cycles in the multigraph. In this game $\delta = 0.5$ is exactly the limit when the dimension jumps up from zero. For example, with $\delta = 0.51$ another cycle appears in node a besides repeating a infinitely. Consequently, $adaaa$ becomes elementary, which makes it possible to play d repeatedly after a as long as at least three a's are played after it. The dimension $s \approx 0.42$ of this two cycle system is computed from equation $0.51^s + 0.51^{5s} = 1$, where five is the length of the new cycle. For $\delta = 0.58$ the multigraph has over one hundred nodes, node a that determines the dimension has over four hundred cycles, and there are over one hundred elementary subpaths with maximal length 22.

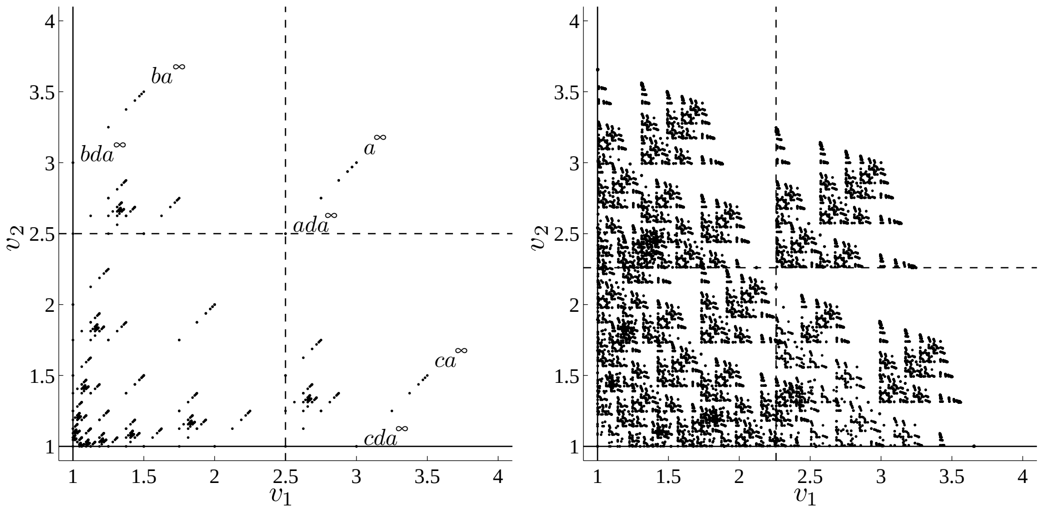


Figure 4. The payoff sets in prisoners' dilemma game for $\delta = 0.5$ and $\delta = 0.58$.

6 Conclusions

This paper both characterizes and offers a way to compute pure strategy subgame perfect equilibrium paths in discounted supergames. Complex strategic behavior is collapsed into subpaths of action profiles that determine the equilibrium outcomes. These subpaths have the property that the players have no incentive to deviate from the first action profile if any equilibrium path follows from the final element of the subpath.

The elementary subpaths form the basis for a recursive presentation of all the equilibrium paths in the game. In particular, using elementary subpaths it is possible to present all equilibrium paths compactly with a multidigraph. Consequently, the set of equilibrium payoffs is a graph-directed self-affine set, when there are finitely many elementary subpaths. More generally the set is sub-self-affine. Due to these observations it is possible to show that when the discount factors are less than one half, the Hausdorff dimension of the payoff set is given by a zero of topological pressure almost surely. This dimension gives a measure for the complexity of the set of equilibrium payoffs. In general, the dimension increases as the discount factors increase.

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